ON A METHOD FOR FINDING THE NUMERICAL SOLUTION OF CAUCHY PROBLEM FOR 2D BURGERS EQUATION

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Abstract

In this paper a new method is proposed for finding the numerical solution of the Cauchy problem for 2D Burgers equation with initial function consisting of four piecewise constants in a class of discontinuous functions. For this goal, a special auxiliary problem which has some advantages over the main problem is introduced. Using these advantages of the auxiliary problem, the numerical solution of the main problem is obtained. Some computer experiments are carried out.

Keywords: Riemann-type problem, 2D-Burgers Equation, Numerical solution in a class of discontinuous functions, Numerical weak solution

Introduction:

As usually, let R^2 be an Euclidean space of points (x, y). Let Q_T be a domain in $R_+^3 = R^2 \times R^+$ as $Q_T = \{(x, y, t) \mid a \le x \le b,\ c \le y \le d,\ 0 \le t < T\}$. Here a, b, c, and T are given constants.

In Q_T, we consider the Riemann-type initial value problem for a two-dimensional scalar equation which describes a certain conservation law as

$$\frac{\partial v(x,y,t)}{\partial t} + v(x,y,t)\frac{\partial v(x,y,t)}{\partial x} + v(x,y,t)\frac{\partial v(x,y,t)}{\partial y} = 0 \qquad (1)$$

with the initial data

$$v(x, y, 0) = v_0(x, y). \qquad (2)$$

Here $v_0(x, y)$ is piecewise constant on a finite numbers of wedges centered at the origin $x = 0, y = 0.$ In particular, what is interesting is the four-wedge problem with wedges corresponding to four quadrants $\left(\alpha_1 = 0, \alpha_2 = \frac{\pi}{2}, \alpha_3 = \pi, \alpha_4 = \frac{3\pi}{2}\right)$ of the spatial plane.

The existence and uniqueness of the solution for the single conservation law in several dimensions by the method of viscosity were studied in [4], [9], [12]. It should be noted that this method gives little information about the qualitative structure of the discontinuity set of solution. This problem for the one space dimension was investigated in [2], [5], [11] etc. where the structure of the solution was revealed in detail.

The solution of problem (1), (2) obtained by using the method of characteristics is

$$v(x, y, t) = v_0(\xi, \eta), \tag{3}$$

where $\xi = x - vt$ and $\eta = y - vt$ are the special coordinates moving with the speed of v, respectively.

First investigations of the two-dimensional Riemann-type initial value problem was initiated by Guckenheimer, [3]. Paper [7] is devoted to a construction of the solution of the 2D- Riemann-type initial value problem for scalar conservation law in the event of a three or more inflection point in state function $f(v) = g(v)$, by analyzing a study of the generalization of one- dimensional Riemann problem to allow for initial data having a finite number of jump discontinuities with constant data or rarefaction waves between jumps. In [6], it has been shown that the solution of the 2D-Riemann-type initial value problem can be classified and present in terms of two-dimensional nonlinear waves in analogy with the nonlinear rarefaction and shock waves of the one dimensional Riemann problem, i.e. explicit solutions are constructible from these waves.

Since the solutions for 2D-Riemann-type initial value problems have explicit structure, they also serve as a touchstone for numerical algorithms. In [1], the concept of the shock and rarefaction base points are included and using the characteristic analysis, the analytical solution for 2D-Riemann-type initial value problem for the Burgers equation is constructed. In addition, for the numerical solution the composite scheme developed by Liska-Wendroff in [8] is applied.

Auxiliary Problems

It is known that global continuous solutions for 2D-Riemann-type initial value problems will not yied appropriate smooth initial data. The weak solution of problem (1), (2), will be defined as follows.

Definition 1: *The function* $v(x, y, t)$ *satisfying initial condition* (*2*) *is called a weak solution of problem* (*1*), (*2*) *if the following integral relation*

$$\iint_{R^2}\int_{R_+^T}\left\{v(x,y,t)\varphi_t(x,y,t)+\frac{v^2(x,y,t)}{2}\varphi_x+\frac{v^2(x,y,t)}{2}\varphi_y(x,y,t)\right\}dxdydt$$
$$+\iint_{R^2}v_0(x,y)\varphi(x,y,0)dxdy=0 \qquad (4)$$

holds for every test function $\varphi(x,y,t)$ defined in R_+^3 and differentiable in the upper half plane and vanishes for $\sqrt{x^2+y^2}+t$ sufficiently large.

Let Q_{xy} be a rectangular domain defined as

$$Q_{xy}=\{(\xi,\eta),\ a\le\xi\le x,\ c\le\eta\le y\}$$

such that $Q_{xy}\subset Q_T$ and $Q_{bd}\times[0,T)=Q_T$.

In order to find the weak solution of problem (1), (2) in the sense of (4) we will introduce the auxiliary problem as above. Integrating equation (1) on the region Q_{xy}, we get

$$\frac{\partial}{\partial t}\int_a^x\int_c^y v(\xi,\eta,t)d\xi d\eta+\frac{1}{2}\int_c^y v^2(x,\eta,t)d\eta+\frac{1}{2}\int_a^x v^2(\xi,y,t)d\xi$$
$$=\varphi_1(a,y,t)+\varphi_2(x,b,t). \qquad (5)$$

Here

$$\varphi_1(a,y,t)=\frac{1}{2}\int_c^y v^2(a,\eta,t)d\eta,\ \ \varphi_2(x,b,t)=\frac{1}{2}\int_a^x v^2(\xi,b,t)d\xi.$$

It is clearly seen that $\varphi_1(a,y,t)+\varphi_2(x,b,t)\in \mathrm{kerM}$. Here $M\{\cdot\}=\frac{\partial^2\{\cdot\}}{\partial x\partial y}$.

We denote by $w(x,y,t)$ the following expression

$$w(x,y,t)=\int_a^x\int_c^y v(\xi,\eta,t)d\xi d\eta+\Psi(a,b,x,y,t), \qquad (6)$$

where $\Psi(a,b,x,y,t)\in kerM$. From (6) it follows that

$$v(x,y,t)=M\{w(x,y,t)\}. \qquad (7)$$

Taking into consideration (6), (7), we get

$$\frac{\partial w(x,y,t)}{\partial t}+\frac{1}{2}\int_c^y v^2(x,\eta,t)d\eta+\frac{1}{2}\int_a^x v^2(\xi,y,t)d\xi=0. \qquad (8)$$

The initial condition for (8) is

$$w(x,y,0)=w_0(x,y). \qquad (9)$$

Here the function $v_0(x,y)$ is any differentiable solution of the equation

$$v_0(x,y)=M\{w_0(x,y)\}. \qquad (10)$$

To find a solution of equation (8) with initial condition (9) we will call a first kind auxiliary problem.

The auxiliary problem (8), (9) has following advantages:

(i) In this case the functions $v(x,y,t)$ and $\frac{1}{2}v^2(x,y,t)$ can be discontinuous too,
(ii) the order of differentiability of the function $w(x,y,t)$ is greater than the order of differentiability of the function $v(x,y,t)$,
(iii) the derivatives v_x, v_y and v_t in algorithm for obtaining of the solution of problem (8), (9) does not occur, as these derivatives does not exist.
The following theorem is valid.
Theorem 1: *If the function $w(x,y,t)$ is a solution of auxiliary problem (8), (9), then the function $v(x,y,t)$ expressed by $v(x,y,t) = Mw(x,y,t)$ is a weak solution of main problem (1), (2).*
To obtain a solution of equation (5) with initial condition (2) we will call second kind auxiliary problem.

Analysis of a Linear Problem

Before obtaining the solution of nonlinear problems, at first we investigate a simple linearized problem (1), (2) as

$$\frac{\partial v(x,y,t)}{\partial t} + A\frac{\partial v(x,y,t)}{\partial x} + B\frac{\partial v(x,y,t)}{\partial y} = 0, \quad (11)$$

where A and B are given constants. The exact solution of (11) with initial condition (2) is

$$v(x,y,t) = v_0(x - At, y - Bt). \quad (12)$$

In this paper we will study the first type auxiliary problem for (11), (12). In this case equation (8) can be rewriten in the following form

$$\frac{\partial w(x,y,t)}{\partial t} + A\int_c^y v(x,\eta,t)d\eta + B\int_a^x v(\xi,y,t)d\xi = 0. \quad (13)$$

For this case, relation (7) is valid too. Taking into consideration (7), we have

$$\frac{\partial w(x,y,t)}{\partial t} + A\frac{\partial w(x,y,t)}{\partial x} + B\frac{\partial w(x,y,t)}{\partial y} = 0. \quad (14)$$

The initial condition for (14) is

$$w(x,y,0) = w_0(x,y). \quad (15)$$

Here $w_0(x,y)$ is any continuous differentiable function of equation (10).
The exact solution of problem (14), (15) is

$$w(x,y,t) = w_0(x - At, y - Bt). \quad (16)$$

It is seen that equation (14) coincides with equation (11), but the initial function $v_0(x,y)$ is more smooth than the initial function of the main problem.

Finite Difference Schemes in a Class of Discontinuous Functions

In this section, we intend to develop the numerical method for finding the solution of problem (1), (2), and investigate some of its properties. As it is stated above, in the nonlinear case, the solution of the main problem has discontinuous points, whose locations are unknown beforehand. These properties do not permit us to apply classical numerical methods to this problem directly. For this aim, we will use auxiliary problem (8), (2). By using the advantages of the suggested auxiliary problem, a new numerical algorithm will be proposed. In [10] the suggested numerical method was studied for two-dimensional nonlinear scalar equation, when $f(v) = g(v) = v^2/2$ and the Riemann data consist of the two segments piecewise constant.

The proposed method will be developed to find the numerical solution of the Cauchy problem for equation (1) in the following study.

The Finite Difference Scheme

In order to develop the numerical algorithm, we construct the following grids. Let

$$\omega_{hx} = \{x_i,\, x_i = a + ih_x,\, h_x = \frac{b-a}{n},\ i = 0,1,2,...,n\}$$

and

$$\omega_{hy} = \{y_j,\, y_j = c + jh_y,\, h_y = \frac{d-c}{m},\ j = 0,1,2,...,m\}$$

which are cover of the segments $[a,b]$ and $[c,d]$ respectively. Now we shall construct two new grids as

$$\omega_{h\xi} = \{\xi_\nu,\, \xi_\nu = a + \nu h_\xi,\, \nu = 0,1,2,...,np\} \text{ and}$$
$$\omega_{h\eta} = \{\eta_\mu,\, \eta_\mu = c + \mu h_\eta,\, \mu = 0,1,2,...,mq\}$$

which also cover the segments $[a,b]$ and $[c,d]$ respectively, where $h_\xi = \frac{hx}{p}$ and $h_\eta = \frac{hy}{q}$. Later we introduce the following notations $\omega_{xy} = \omega_{hx} \times \omega_{hy}$, $\omega_{\xi\eta} = \omega_{h\xi} \times \Omega_{h\eta}$, $\omega_{xy}^T = \omega_{xy} \times \{t_k = k\tau;\, k = 0,1,2,...\}$ and $\omega_{\xi\eta}^T = \omega_{\xi\eta} \times \{t_k = k\tau;\, k = 0,1,2,...\}$. Since $\xi_{p\ell} = x_\ell$, $\eta_{q\lambda} = y_\lambda$ for any $0 \le \ell \le n$ and $0 \le \lambda \le m$ it clear that $\omega_{xy}^T \subseteq \omega_{\xi\eta}^T$. Here n, m, p, q are the given integer numbers which show nodal points on the segments $[a,b]$, $[c,d]$, $[a,x]$ and $[c,y]$ respectively.

In order to approximate equations (8) (or (13)) by finite difference, integrals leaving in (8) (or (13)) are approximated as follows:

$$\int_a^{x_i} v(\xi, y, t)d\xi = h_\xi \sum_{\nu=1}^{pi} V(\xi_\nu, y_j, t_k), \qquad (17)$$

$$\int_c^{y_j} v(x, \eta, t)d\eta = h_\eta \sum_{\mu=1}^{qj} V(x_i, \eta_\mu, t_k), \qquad (18)$$

$$(i = 1,2,...,n), \quad (j = 1,2,...,m).$$

Taking into consideration (17) and (18), equations (8) and (13) at any point (i, j, k) of the grid ω_{xy}^T are approximated by the following explicit scheme as

$$W_{i,k+1} = W_{i,k} - \frac{h_t h_\eta}{2} \sum_{\mu=1}^{qj} V^2(x_i, \eta_\mu, t_k) - \frac{h_t h_\xi}{2} \sum_{\nu=1}^{pi} V^2(\xi_\nu, y_j, t_k), \qquad (19)$$

$$W_{i,k+1} = W_{i,k} - Ah_t h_\eta \sum_{\mu=1}^{qj} V(x_i, \eta_\mu, t_k) - Bh_t h_\xi \sum_{\nu=1}^{pi} V(\xi_\nu, y_j, t_k), \qquad (20)$$

$$(i = 1,2,...,n; \ j = 1,2,...,m; \ k = 0,1,2,...).$$

Computer Experiments

Three type computer tests on basis of the proposed algorithms are carried out. Tests were made using the following data: $(x, y) \in (-2,2) \times (-2,2), T = 0.2, h_t = 0.001$, $n = m = 500$ and $(v_1, v_2, v_3, v_4) = (1, 2, 3, 4)$.

At first we consider the initial value problem for (11) with piecewise constants connecting four wedges centered at the origin $x = 0$, $y = 0$, i.e

$$v_0(x, y) = \begin{cases} v_1, & x > 0, y > 0, \\ v_2, & x < 0, y > 0, \\ v_3, & x < 0, y < 0, \\ v_4, & x > 0, y < 0. \end{cases} \qquad (21)$$

We will solve problem (14), (15) instead of (11), (2). In this case the function of $w_0(x, y)$ will be chosen as a continuous solution of (10)

$$w_0(x, y) = \begin{cases} v_1 xy, & x > 0, y > 0, \\ v_2 xy, & x < 0, y > 0, \\ v_3 xy, & x < 0, y < 0, \\ v_4 xy, & x > 0, y < 0. \end{cases} \qquad (22)$$

Problem (14), (15) is approximated by following explicit finite difference schemes as

$$W_{i,j,k+1} = g_1 W_{i-1,j,k} + [1 - (g_1 + g_2)]W_{i,j,k} + g_1 W_{i,j-1,k}, \qquad (23)$$

$$W_{i,j,0} = w_0(x_i, y_j). \qquad (24)$$

Here $g_1 = \frac{h_t A}{h_x}$, $g_2 = \frac{h_t B}{h_y}$ and $W_{i,j,k+1}$ denote the approximated value of $w(x, y, t)$ at any point (x_i, y_j, t_k) of the grid $\omega_{xy}^T \subseteq \omega_{\xi\eta}^T$. It is no difficult to indicate that

$$V_{i,j,k+1} = \frac{1}{h_x}\left\{\frac{W_{i-1,j,k} - W_{i-1,j,k}}{h_y} - \frac{W_{i-1,j,k} - W_{i-1,j,k}}{h_y}\right\}.$$

Second type calculation have been maked on basis of the following

$$V_{i,j,k+1} = V_{i,j,k} - \frac{h_t}{h_x h_y}\left[(h_x + h_y)V_{i,j,k}^2 + h_y V_{i-1,j,k}^2 + h_x V_{i,j-1,k}^2\right],$$

$$(i = 1,2,...,n),\ j = 1,2,...,m,\quad k = 0,1,2,...,),$$

$$V_{i,j,0} = v_0(x_i, y_j),\quad (i = 0,1,2,...,n,\quad j = 0,1,2,...,m)$$

difference schemes.

Finaly, using difference schemes (19), (20) the numerical solution of the problems (1),(2) and (11),(2) on the same data are found. Obtained solutions show good coincidence with exact solutions of the investigated problem.

References:

Daeki Yoon, Woonjae Hwang, Two-Dimensional Riemann Problems for Burger's Equation, Bull. Korean Math. Sos. 45, (2008), No. 1, pp. 191-205.

Gelfand, I., M., Some problems in the theory of quasilinear equations. Usp Math. Nauk, 14, 87-158 (1959), Amer. Math. soc Transl. (2) 29, 295-381 (1963).

Guckenheimer, J. Shocks and Rarefactions in Two Space Dimensions, Arch. Rational Mech. Anal. 59 (1975), No. 3, 281-291.

Kruzkov, S.N., First Order Quasilinear Equation in Several Independent Variables, Math. Sbornik, 81, 1970, p. 217-243.

Lax, P.D., Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves. Conf. Board Math. Sci. Regional Conf. Series in Appl. Math. 11, SIAM, Philadelphia, 1972.

Lindquist, W.B., The Scalar Riemann Problem in Two Spatial Dimensions: Piecewise Smoothness of Solutions and Its Breakdown, SIAM J. Math. Anal., 17 (1986), 1178-1197.

Lindquist, W.B., Construction of Solutions for Two-Dimensional Riemann Problems, Comput. Math. Appl. 12A (1986), 615-630.

Liska, R., Wendorff, B., Composite Schemes for Conservation Laws, SIAM J. Num. Anal., 35 (1998), 2250-2271.

Oleinik, O.A., Discontinuous Solutions of Nonlinear Differential Equations, Usp.Math. Nauk, 12, 1957.

Rasulov, M.A., Coskun, E., Sinsoysal, B., Finite Differences Method for a Two-Dimensional Nonlinear Hyperbolic Equations in a Class of Discontinuous Functions. App. Mathematics and Computation, vol.140, Issue l, August, pp.279-295, 2003, USA.

Smoller, J.A. Shock Wave and Reaction Diffusion Equations, Springer-Verlag, New York Inc., 1983.

Volpert, A.I. The Spaces BV and Quasilinear Equations. Math. USSR-Sbornik 2, 225-267, 1967.