VARIANCE COMPONENTS ESTIMATION IN A K-WAY NESTED RANDOM EFFECT MODELS

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Abstract
This work aims to presented the k - way nested random models (designs) and discuss the estimators for the variance components in this kind of models presented by Henderson (1953), proposing condition concern their existence, as well as its two proposed modifications presented by Khattree (1999). At the first proposal, where he choose to sacrifice the unbiasedness of the Henderson's one to preserve the nonnegativity of the variance, and after noting (through simulations) that his estimator has better performance than the Henderson's one except for the variance of the error term, which has less mean square error than the corresponding error term of the Khattree's estimator, Khattree (1999), on is proposed modification, replaced the error term by the error term of the Henderson's estimators. Using Henderson's estimators, method of determining hypothesis tests based on Satterwaite (1946) procedure are discussed as well.

Keywords: Nested random models, variance components, henderson's estimators, khattree's estimators, tests of hypothesis

Introduction
The completed nested models arise in many experiments and surveys. Suppose, for sake of motivation, that some local government is interested in choose laboratories to adminstrates urinalyses to human subject, who work caring for plants with hazardous materials, and nextly several analyses may be made from each urine samples, so that the underlying model involves laboratories, human subjects within the laboratories, tests within human subjects, and analysis with tests. Each one of the factor (stage) may be chooses fixed or in a random way. For a variety of reasons, more sublevels (data) may be available for some levels than for others, i.e., the data are unbalanced. For example, there may be some occasions on which some subjects does not appears to his test, or more analysis is needed at different levels of the factor tests, so that the sublevels within each level may varying. In this case the coefficient of a particular variance component in the mean squares expectation will vary from one mean square for another, leading to strong difficulties in computuation of the variance components estimations or in performance of tests of hypothesis, situation which does not holds for the case when the data are balanced (see Gates and Shiue (1962)).

This work aims to analyses the estimators for the variance components proposed separately by Henderson (1953) and Khattree (1999) in a k - way nested random model (see, for instance, Gates and Shiue (1962) for notions of this kind of model), proposing condition on their existence, and the performance of tests of hypothesis for the variance components, taking the data to be unbalanced and assuming that the observations of the last factor levels, which are randomly taken, constitute the (k-1)th factor (see Tietjen and Moore Tietjen (1968) for tests of hypothesis in k - way random nested models). The approach to the construction of the estimators for variance components presented here is proposed by Henderson (1953).
General K-Way Nested Model - Basic Notions

A statistical model is said be a k-way nested one if it consists of k factors, say $A_1, \ldots, A_k$ (see the Figure 1), having each one of than some levels, where:

- The levels of the factor $A_k$ are nested within the levels of the factor $A_{k-1}$;
- The levels of the factor $A_{k-1}$ are nested within the levels of the factor $A_{k-2}$;
  
  
  - The levels of the factor $A_{k-3}$ are nested within the levels of the factor $A_{k-2}$;
- The levels of the factor $A_{k-2}$ are nested within the levels of the factor $A_{k-1}$.

The effect associated with any factor is the effect which its levels have on the interest response variable.

One supposes now that:

There is $a_{(k+1)}j_{h_1}j_{h_2} \ldots j_{h_k}$ observations nested within the $j_k$th level of the factor $A_k$;

The factor $A_k$ has $a_{k(j_1 \ldots j_{k-1})}$ levels nested within the $j_{k-1}$th level of the factor $A_{k-1}$ ($j_{k-1} = 1, \ldots, a_{(k-1)}j_{h_{k-2}} \ldots j_{h_1}$);

The factor $A_{k-1}$ has $a_{(k-1)}j_{h_{k-3}} \ldots j_{h_2}$ levels nested within the $j_{k-2}$th level of the factor $A_{k-2}$ ($j_{k-2} = 1, \ldots, a_{(k-2)}j_{h_{k-4}} \ldots j_{h_2}$);

- The factor $A_{j_{h_{k-1}}}$ has $a_{j_{h_{k-1}}}$ levels nested within the $j_{k-1}$th level of the factor $A_{j_{h_{k-2}}}$ ($j_{h_{k-1}} = 1, \ldots, a_{j_{h_{k-2}}}$);
- The factor $A_{j_{h_{k-2}}}$ has $a_{j_{h_{k-2}}}$ levels nested within the $j_{h_{k-2}}$th level of the factor $A_{j_{h_{k-3}}}$ ($j_{h_{k-2}} = 1, \ldots, a_{j_{h_{k-3}}}$);

The factor $A_1$ has $a_{1}$ levels.

Figure 1: k-way nested model with unbalanced data, with factors $A_1, \ldots, A_k$. The observations (sublevels) nested within the different levels of the factor $A_k$ are assumed to constitute the factor $A_{k+1}$.

So, a k-way nested random model can be written as:

$$y_{j_{h_1} \ldots j_{h_k}} = \mu + \beta_{j_h} + \beta_{j_{h_2}(j_h)} + \beta_{j_{h_3}(j_h)} + \cdots + \beta_{j_{h_{k-3}}(j_h)} + \beta_{j_{h_{k-1}}(j_h)}$$

$$ = \mu + \beta_{j_h} + \sum_{i=2}^{k-1} \beta_{j_{h_i}(j_h)} \quad (0.1)$$

with
\[
\begin{align*}
J_i &= 1, \ldots, a_i, \\
J_j &= 1, \ldots, a_{j_{k+1}}, \ i = 2, \ldots, k + 1,
\end{align*}
\]

where

\[y_{j_1, \ldots, j_k, i} \text{ is the } j_{k+1}^{th} \text{ observation of the } j_i^{th} \text{ level of the factor } A_i \text{ nested within the } j_{k+1}^{th} \text{ level of the factor } A_{k-1} \text{ nested within the } j_{k+2}^{th} \text{ level of the factor } A_{k-2} \text{ nested } \ldots \text{ nested within the } j_{k}^{th} \text{ level of the factor } A_2 \text{ nested within the } j_{k+1}^{th} \text{ level of the factor } A_1; \]

\[\mu \text{ represents the general mean; } \]

\[\beta_{j_1} \text{ is the random effect due to the } j_1^{th} \text{ level of the factor } A_1; \]

\[\beta_{j_2(j_1)} \text{ is the random effect due to the } j_2^{th} \text{ level of the factor } A_2 \text{ nested within the } j_1^{th} \text{ level of the factor } A_1; \]

\[\beta_{j_2(j_3, j_4)} \text{ is the random effect due to the } j_3^{th} \text{ level of the factor } A_3 \text{ nested within the } j_2^{th} \text{ level of the factor } A_2 \text{ nested within the } j_1^{th} \text{ level of the factor } A_1; \]

\[\cdots \cdots \]

\[\beta_{j_{k+1}(j_{k+2}, \ldots, j_k)} \text{ is the random effect due to the observation } y_{j_{k+1}, \ldots, j_k}. \]

Following Sahai and Ojeda (2005), one assumes the \(\beta's\) to be mutually and completely uncorrelated variables with means zero and variance \(Var(\beta_{j_i}) = \sigma^2_i, \)

\[Var(\beta_{j_{k+1}, \ldots, j_k}) = \sigma^2, \]

\[Var(\beta_{j_{k+1}, j_{k+2}, \ldots, j_k}) = \sigma^2_i, \quad i = 2, \ldots, k, \quad \text{and } Var(\beta_{j_{k+1}, j_{k+2}, \ldots, j_k}) = \sigma^2_e. \]

This last one is the variance of the error term. Here, \(\sigma^2_1, \ldots, \sigma^2_k, \sigma^2_e, \sigma^2 \) and \(\sigma^2_e, \sigma^2 \) are known as the variance components of the response variable.

**The Analysis of Variance: The Expected Mean Squares**

In order to establish the analysis of variance (ANOVA) sum of squares for each factor, one firstly provides the sums of the number of levels for different factors, as well as the sums of observations at different levels.

Conveniently, one denotes the number of levels at factor \(A_i\) as

\[a_{j_i} = \sum_{j_{k+1}} \sum_{j_k} a_{j_{k+1}, j_{k+2}, \ldots, j_k}, \ i = 1, \ldots, k - 1, \]

\[a_{j_i} = a_{j_{k+1}, j_{k+2}, \ldots, j_k}, \]

and the total number of levels (observations) in the sample by \(a_{j_i} = \sum_j a_{j_i}.\)

The general sum of observations, denoted by \(y_{j_0},\) is

\[y_{j_0} = \sum_j \sum_{j_{k+1}} y_{j_{k+1}, \ldots, j_k}, \]

and the sums of observations at different levels is given by:

\[y_i = \sum_{j_i} \sum_{j_{k+1}} y_{j_{k+1}, \ldots, j_k}, \ i = 1, \ldots, k. \quad (0.2)\]

Thus, the sum of squares for the factor \(A_i\) is given by
\[ SS_A = \sum \frac{y_{ij}^2}{a_{ij}} - \bar{a}_{i} \]  \hspace{1cm} (0.3)

The one for the remaining factors \( A_i, \ i = 1, \ldots, k, \), by
\[ SS_A = \sum \cdots \sum \frac{y_{ijk}^2}{a_{k}} - \sum \cdots \sum \frac{y_{ijk}^2}{a_{j}} = \sum \cdots \sum \frac{y_{ijk}^2}{a_{k}} - \sum \cdots \sum \frac{y_{ijk}^2}{a_{j}} \]  \hspace{1cm} (0.4)

and the sum of squares for the errors is given by
\[ SS_e = \sum \cdots \sum \frac{y_{ijk}^2}{a_{k}} - \sum \cdots \sum \frac{y_{ijk}^2}{a_{j}} \]

See Sahai and Ojeda (2005) for some additional explanation.

Having in mind the total number of observations at different factors \( A_i, \ i = 1, \ldots, k, \) the degrees of freedom, \( d_i, \ i = 1, \ldots, k + 1, \) at each sources of variation is computed as follows:
\[ d_1 = a_1 - 1; \quad d_2 = \sum_{i, j} a_{2(i,j)} - a_1; \]
\[ d_i = \sum_{j, k} \left( \sum_{j, k} a_{(i,j,k)} - a_{(i,k)} \right), \quad i = 3, \ldots, k + 1; \]  \hspace{1cm} (0.5)

is the degrees of freedom among the levels of the factor \( A_i, \ d_i, \ i = 1, \ldots, k, \) the degrees of freedom among levels of the factor \( A_i \) nested within the factor \( A_{i-1}, \) and \( d_{k+1} \) the one among the error factor.

Thus, the mean square (which is obtained by dividing the sum of squares by its corresponding degrees of freedom), denoted here by \( MS_A, \ i = 1, \ldots, k + 1, \) can be written as
\[ MS_A = \frac{SS_A}{d_i}, \ i = 1, \ldots, k, \]
and
\[ MS_{A, i} = \frac{SS_A}{d_{k+1}}. \]

Now, in what follows, one presents the result concerning expected mean square. Such result, which is presented here as a proposition, can be found at Sahai and Ojeda (2005) or, for instance, at Searle at al. (2006).

**Proposition 1.**

Consider all the results established up to now. Then, the expected mean square at each source of variation is given by
\[ E(MS_A) = EMS_A = \begin{cases} \sigma_e^2 + c_{i, s} \sigma_e^2 + c_{i, k+1} \sigma_e^2 + \cdots + c_{i, s} \sigma_e^2, & i = 1, \ldots, k, \\ \sigma_e^2, & i = k + 1, \end{cases} \]  \hspace{1cm} (0.6)

where \( c_{i, s}, \ i \leq s \leq k, \) are given by
\[ c_{i, s} = \sum_{j, k} \left( \sum_{j, k} a_{(i, j, k)} - a_{(i, k)} \right) \frac{1}{d_i} \]  \hspace{1cm} (0.7)

The proof of the above result is very tedious and expensive in what concern the time to perform it, so one will not give it. Instead of that, one recommends Gates and Shiue (1962) or Sahai and Ojeda (2005) for some more details.

**Note 1.**
It must be noted that the condition $a_{ij} \neq 0 \neq a_{ss}$, $i = 1, \ldots, k$, which means every factor has always at least one level, must hold in order to ensure the existence of the $c_{i,s}$, $i \leq s \leq k$.

The system of equation (1.6) can be rewritten in the matrix notation as follows:

\[
[EMS_{A} EMS_{A} : EMS_{A} EMS_{A}] =
\begin{bmatrix}
1 & c_{1,k} & c_{1,k-1} & \cdots & c_{1,3} & c_{1,1} \\
1 & c_{2,k} & c_{2,k-1} & \cdots & c_{2,3} & c_{2,2} & 0 \\
1 & c_{3,k} & c_{3,k-1} & \cdots & c_{3,3} & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & c_{k-1,k} & c_{k-1,k-1} & \cdots & 0 & 0 & 0 \\
1 & c_{k,k} & 0 & \cdots & 0 & 0 & 0 \\
1 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\times
\begin{bmatrix}
\sigma^2_c \\
\sigma^2_s \\
\sigma^2_k \\
\sigma^2_{k-1} \\
\sigma^2_j \\
\sigma^2_{i,s}
\end{bmatrix}
\]

(0.8)

**Estimation of the Variance Components**

One of the most common procedure for the *estimation of variance components* is the one suggested by Henderson (1953) through is three variations known as *method 1*, *method 2*, and *method 3*, especially because of its simplicity in what concern the computational implementation (even on a hand-held calculator), and unbiasedness. Although the three methods common underlying idea is to form the (observed) *mean squares* for different factors and then equating them to their respective *expected mean squares* (in some case with some readjustment), leading to a system of linear equations, which solved in the *variance components* gives the corresponding estimator, the scenario in this paper is appropriate for the *method 1*, since all terms $\beta_{j}$ and $\beta_{j(k-k+1)}$, $i = 2, \ldots, k + 1$, are regarded as random variables. Such estimators, as well as their existence and consistence, are discussed at the subsection which comes next (Subsection 3.1).

Despite its good performance, the Henderson's estimators for the *variance components* abdicate the nonnegativity of variance, situation which is approached by Khattree (1999) on its proposed estimators. The approach proposed by Khattree (1999) which preserve the nonnegativity of variance constitutes a modification to the Henderson's estimators. This is discussed at Subsection 3.2.

**Estimators Proposed By Henderson**

On this subsection one will present the estimator proposed by Henderson (1953) (making use of its method 1) to obtain the estimators for the *variance components* in models with (completely) random designs, which is the case of the one discussed here (see model (1.1)), and proposes some additional condition over that model in order to get the desired estimators.

Recall the result concerning the *expected mean squares* (the system of equations (1.6) or its matricial notation (1.8)).

Rearranging the matrices involved in (1.8), such result can be, equivalently, rewritten as follows:
\[
\begin{bmatrix}
\text{EMS}_{A_1} \\
\text{EMS}_{A_2} \\
\text{EMS}_{A_3} \\
\vdots \\
\text{EMS}_{A_{k-1}} \\
\text{EMS}_{A_k} \\
\text{EMS}_{A_{k+1}}
\end{bmatrix}
= 
\begin{bmatrix}
c_{1,1} & c_{1,2} & c_{1,3} & \cdots & c_{1,k-1} & c_{1,k} \\
0 & c_{2,2} & c_{2,3} & \cdots & c_{2,k-1} & c_{2,k} \\
0 & 0 & c_{3,3} & \cdots & c_{3,k-1} & c_{3,k} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & c_{k-1,k-1} & c_{k-1,k} \\
0 & 0 & 0 & \cdots & 0 & c_{k,k} \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\times
\begin{bmatrix}
\sigma_1^2 \\
\sigma_2^2 \\
\sigma_3^2 \\
\vdots \\
\sigma_{k-1}^2 \\
\sigma_k^2 \\
\sigma_e^2
\end{bmatrix}
\tag{0.9}
\]

Now, let
\[
M = \begin{bmatrix}
\text{MS}_{A_1} & \text{MS}_{A_2} & \text{MS}_{A_3} & \ldots & \text{MS}_{A_{k-1}} & \text{MS}_{A_k} & \text{MS}_{A_{k+1}}
\end{bmatrix}
\]

be the vector whose the entries are the mean squares of the different factors, and
\[
E = \begin{bmatrix}
\text{EMS}_{A_1} & \text{EMS}_{A_2} & \text{EMS}_{A_3} & \ldots & \text{EMS}_{A_{k-1}} & \text{EMS}_{A_k} & \text{EMS}_{A_{k+1}}
\end{bmatrix}
\]

the vector whose the entries are the expected mean squares of the different factors, and
\[
\sigma = [\sigma_1^2 \sigma_2^2 \sigma_3^2 \ldots \sigma_{k-1}^2 \sigma_k^2 \sigma_e^2]
\]

the vector of the variances of the effects due to different factors (including the one for the error term). In order to find the estimates for the variance components, one must solves the following system of linear equation in \(\sigma\):
\[
E(M) = E = C\sigma, \tag{0.10}
\]

where
\[
C = \begin{bmatrix}
c_{1,1} & c_{1,2} & c_{1,3} & \cdots & c_{1,k-1} & c_{1,k} \\
0 & c_{2,2} & c_{2,3} & \cdots & c_{2,k-1} & c_{2,k} \\
0 & 0 & c_{3,3} & \cdots & c_{3,k-1} & c_{3,k} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & c_{k-1,k-1} & c_{k-1,k} \\
0 & 0 & 0 & \cdots & 0 & c_{k,k} \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\tag{0.11}
\]

The system (1.10) yields the (unique) consistent solution
\[
\hat{\sigma} = C^{-1}M ,
\]
with \(\hat{\sigma} = \begin{bmatrix}
\sigma_1^2 & \sigma_2^2 & \sigma_3^2 & \ldots & \sigma_{k-1}^2 & \sigma_k^2 & \sigma_e^2
\end{bmatrix}^T\), if the matrix \(C\) is nonsingular (invertible), having, therefore, the inverse \(C^{-1}\). So one has only to ensure the nonsingularity of the matrix \(C\).

**Note 2.**

One should note that, in fact, the Henderson's estimator is unbiased. Indeed,
\[
E(\hat{\sigma}) = C^{-1}E(M) = C^{-1}C\sigma = \sigma.
\]

**Proposition 2.**

The necessary and sufficient condition for the matrix \(C\) in the equation (1.10) to be nonsingular is
\[a_{i-1} \neq a_i, i = 1, \ldots, k.\]
Proof. Since the matrix $C$ is a triangular one, it is nonsingular if and only if all its diagonal elements are nonzero (this is ensured by the Theorem of Lipschutz (1991)). But this holds if and only if
\[ c_{ij} = \sum_{k} \cdots \sum_{k} \left[ a_{ii} - \frac{a_{ij}}{a_{ii-1}} \right] \frac{1}{d_i} \neq 0, \quad i = 1, \ldots, k, \]
which, by its turn, holds if and only if $a_{i-1, i} \neq a_{ii}, a_{ii} \neq 0 \neq a_{ii}, i = 1, \ldots, k$. See Note 1 for the latter inequality explanation.

Thus, in order to guarantee that the condition $a_{i-1, i} \neq a_{ii}, i = 1, \ldots, k$, holds, and, therefore, the existence of $C^{-1}$, it must be assumed that each factor $A_i, i = 1, \ldots, k$, has more than one level and each level must have some (sub) levels nested within. One should note that the levels nested within the levels of the factor $A_i$ are, clearly, the observations.

In the next subsection it is discussed the notable modification to the Henderson's estimators proposed by Khattree (1999).

Estimators proposed by Khattree

As seen at the preceding sections and subsections, the Henderson's estimator $\hat{\sigma} = C^{-1}M$ which is a consistent solution to the system $SM = C\sigma$ do not preserve the nonnegativity of the variance, preserving instead the unbiasedness. On its suggested modification to Henderson's estimator, Khattree (1999) (see Khattree (1998) as well), unlike Henderson (1953), choose to sacrifice the unbiasedness of its components to guarantee their nonnegativity.

Namely, on its approach, he considered the problem:
\[ \min_{\sigma \geq 0} \| M - E(M) \| = \min_{\sigma \geq 0} \| M - C\sigma \|, \quad (0.12) \]
with $M$, $C$ and $\sigma$ defined above, and $\| \|$ (although it may be an appropriate norm) taken to be the Euclidean norm $\| \| = (x^T x)^{1/2}$.

In an equivalent way, the problem (1.12) can be stated as
\[ \min_{\sigma \geq 0} \frac{1}{2} \sigma^T C^T C \sigma, \quad (0.13) \]
which is a quadratic problem with linear constrains $\sigma \geq 0$. Such problem, as did Lemke (1962), using the primal-dual notion relationship of (1.13) with another optimization problem, in our case can be posed in a different way: find $\sigma$ and $u$ such that
\[ \sigma - (C^T C)^{-1} u = \hat{\sigma}, \sigma, u \geq 0, \sigma^T u = 0, \quad (0.14) \]
with $\hat{\sigma}$ the Henderson's estimator (see subsection 3.1).

Let $\hat{\sigma} = \left[ \sigma_1^2, \sigma_2^2, \sigma_3^2, \ldots, \sigma_{k-1}^2, \sigma_k^2, \sigma^2 \right]$ be such solution on $\sigma$.

Through simulation using a $S3S$-way nested random model, Khattree (1998) remarked (Khattree and Gill (1988) and Ahrens at al (1981) made the same remarks, although in others contexts) that the mean square error of its suggested variances components estimators $\sigma_i^2$ is generally smaller than the correspondent one of the estimator $\hat{\sigma}$ suggested by Henderson, $\hat{\sigma}_i^2$, except for the case of the error variance component estimator $\hat{\sigma}_e^2$. For that case he remarked that the result
\[
E \left[ \left( \sigma^2_\varepsilon - \bar{\sigma}^2_\varepsilon \right)^2 \right] \leq E \left[ \left( \sigma^2_\varepsilon - \sigma^2_\epsilon \right)^2 \right]
\]

holds in generally, that is \( \bar{\sigma}^2_\varepsilon \) has in generally less mean square error than \( \sigma^2_\epsilon \).

In order to get a estimator with higher performance, Khattree (1998) (see Khattree (1999) as well) combines the “goodness” of his suggested one with the “goodness” of the one suggested by Henderson (1953) performing a modification which applied to our case can be stated as follow.

Let \( \sigma_1(\tau) = \left[ \sigma^2_1, \sigma^2_2, \sigma^2_3, \ldots, \sigma^2_{k-1}, \sigma^2_k \right] \)”, that is the \( k \) first variance components (dismissing the one of the error term), \( M_1 = \left[ MS_{A_1}, MS_{A_2}, \ldots, MS_{A_{k-1}}, MS_{A_k} \right] \), the mean square of the different first \( k \) factors, and

\[
C = \begin{bmatrix} C_{(11)} & 1 \\ 0 & 1 \end{bmatrix}, \quad (0.15)
\]

where \( I \) is an unitary vector with dimension equal to the row number of the sub matrix \( C_{(11)} \), so that

\[
\sigma = \begin{bmatrix} \sigma_1(\tau) & \sigma^2_\varepsilon \end{bmatrix}, \quad M = \begin{bmatrix} M_1 & MS_{A_{k-1}} \end{bmatrix},
\]

and once the system of equation (1.10) can be rewritten as

\[
M_1 = C_{(11)} \sigma_1(\tau) + \sigma^2_\varepsilon,
\]

\[
MS_{A_{k-1}} = \sigma^2_\varepsilon,
\]

equivalently,

\[
M_1 - MS_{A_{k-1}} = C_{(11)} \sigma_1(\tau) + \sigma^2_\varepsilon,
\]

the problem (1.12) can now be stated as

\[
\min_{\sigma \geq 0} \| M^* = C_{(11)} \sigma_1(\tau) \|, \quad (0.16)
\]

with \( M^* = M_1 - MS_{A_{k-1}} \), which amounts to find (following Lemke (1962)) \( \sigma_1(\tau) \) and \( s \) such that

\[
\sigma_1(\tau) - (C_{(11)} C_{(11)})^{-1} s = \bar{\sigma}_1(\tau), \quad \sigma_1(\tau), s \geq 0, \sigma_1(\tau)^t s = 0. \quad (0.17)
\]

Let \( \bar{\sigma}_1(\tau) \) be the solution in \( \sigma_1(\tau) \). Thereby the Khattree suggested estimator for the variance components is then

\[
\tilde{\sigma} = \begin{bmatrix} \bar{\sigma}_1(\tau) & \bar{\sigma}^2_\varepsilon \end{bmatrix},
\]

with \( \bar{\sigma}^2_\varepsilon \) the Henderson's estimator for the error term.

**Hypothesis Tests for the Variance Components**

Tietjen and Moore (1968) proposed a method to constructing an approximate (pseudo) F-tests to test the hypothesis

\[
H_0^r : \sigma^2_\varepsilon = 0 \quad \text{vs} \quad H_1^r : \sigma^2_\varepsilon > 0, \quad r \in \{1, \ldots, k+1\} \quad (0.18)
\]

in nested models with unbalanced data, based on Satterwaite (1946) procedure for testing a linear combination of the mean squares. To perform such tests one construct the ratio \( \frac{MS_{A_r}}{D_r} \),
where \( MS_A \) is the mean squares error at the factor \( r \), having expectation
\[
E[MS_A] = c_{r,r} \sigma^2 + c_{r, r+1}^2 \sigma^2_{r+1} + \ldots + c_{r, k-1}^2 \sigma^2_{k-1} + c_{r,k}^2 \sigma^2_k + \sigma^2_e.
\]

and \( D_r = \sum_{i=r+1}^{k+1} l_i MS_A \) an appropriate linear combination (presented below: equation (1.20)) of the mean squares

\( MS_{A,i}, \ldots, MS_{A,k} \),

having expected value, i.e., \( E(D_r) \), given by
\[
E(D_r) = c_{r,r}^2 \sigma^2 + c_{r, r+2}^2 \sigma^2_{r+2} + \ldots + c_{r, k-1}^2 \sigma^2_{k-1} + c_{r,k}^2 \sigma^2_k + \sigma^2_e
\]

which has an approximate \( F \) distribution with appropriate degrees of freedom. The degree of freedom of the numerator is \( d_r \) (see result (1.5)). To calculate those for the denominator, considering \( C_r \) the \( r \)th row of the matrix \( C \) (see the matrix \( C \) in (1.11)), clearly \( EMS_A = C_r \sigma \), so that
\[
D_r = C_r \hat{\sigma} - c_{r,r} \hat{\sigma}_r^2 = MS_A - c_{r,r} \hat{\sigma}_r^2
\]
is the desired denominator of the ratio above.

The degrees of freedom for the denominator are usually given by
\[
d_r^* = \left( \frac{\sum_{i=r+1}^{k+1} l_i MS_A} {\sum_{i=r+1}^{k+1} \left( l_i MS_A \right)^2} \right)^2.
\]

Let assume now that \( \sigma_{k+1}^2 = \sigma^2_e \) and \( \hat{\sigma}_{k+1}^2 = \sigma_e^2 \).

Taking \( c_{i,j}^* \) to be the \( i \)th row and \( j \)th column element of the matrix \( C^{-1} \), i.e., the inverse matrix of the matrix \( C \), noting that
\[
D_r = c_{r,r} c_{r,r} \hat{\sigma}_r^2 + c_{r, r+2} \hat{\sigma}_r^2 + \ldots + c_{r, k-1} \hat{\sigma}_k^2 + c_{r,k} \hat{\sigma}_k^2 + \hat{\sigma}_e^2
\]

\[
= c_{r,r} \sum_{j=1}^{k+1} c_{i,j}^* MS_{A,j} + c_{r, r+2} \sum_{j=1}^{k+1} c_{i,j}^* MS_{A,j+2} + \ldots + c_{r, k-1} \sum_{j=1}^{k+1} c_{i,j}^* MS_{A,k-1} + c_{r,k} \sum_{j=1}^{k+1} c_{i,j}^* MS_{A,k} + \sum_{j=1}^{k+1} c_{i,j}^* MS_{A,j},
\]

(0.20)

where \( \hat{\sigma}_i^2 \), \( i, \ldots, k+1 \), are the Henderson variance components estimators, which are unbiased as showed the Note 2, reorganizing the term on (1.20) and noting that \( \sum_{i=r+1}^{k+1} c_{r,i} c_{i,j}^* \) is the coefficient of \( MS_{A,j} \), and also that except for the absence of the nonzero term \( c_{r,r} c_{r,j}^* \) (and the \( r-1 \) terms each equal to zero), the expression \( \sum_{i=r+1}^{k+1} c_{r,i} c_{i,j}^* \) is recognized to be the element at the \( r \)th row and \( j \)th column of \( CC^{-1} = I \), where \( I \) is the identity matrix of order \( k+1 \).

Thus, by adding and subtracting the term \( c_{r,r} c_{r,j}^* \) to the expression of \( D_r \), one obtain
\[
D_k = -\sum_{i=r+1}^{k+1} c_{r,i} c_{i,j}^* MS_A.
\]

(0.21)

The coefficient of \( MS_A \) is zero since the diagonal elements of \( C^{-1} \) are the reciprocals of the diagonal of \( C \). Therefore, the degrees of freedom for the denominator are given by
\[ d_r^* = \frac{D_r^2}{\sum_{i=r+1}^{k+1} \left( \frac{c_{i,r} c_{r,r}^* MS_{i,r}}{d_i} \right)^2}. \] (0.22)

This last expression of \( d_r^* \) is easily computed provided the matrix \( C \) and its inverse \( C^{-1} \).

So, the test procedure for testing the hypothesis \( H_o^r \) vs \( H_i^r \) is based on ratio statistic \( \frac{MS_{i,r}}{D_r} \), which follow an approximate \( F \) - distribution with \( d_r \) and \( d_r^* \) degrees of freedom.

**Conclusion**

The both Henderson's estimator and Khattree's estimators for the variance components discussed here should not be seen mutually exclusive, seen instead to complement each other, in the sense that when the data are note highly unbalanced or when certain condition on variance components are not violated (the nonnegativity of the variance, for example), the method to find estimators suggested by Henderson (1953) are adequate, as showed Swallow and Monahan (1984) (among others) through Monte Carlo Simulation. Indeed, the underlying approach to his method still play a central role in the variances approaches since it is never totally dismissed, been, instead of, appropriately modified by many researchers (Blackwell at al.\cite{Blackwell(1991)} proposed to assign zero to the values of the estimators when the correspondent estimative are negative, but this, clearly, compromise their weak optimality of the unbiasedness). On the other hand, the Khattree's estimator and the Khattree's modificate estimator, which as seen constitute a modification to the Henderson's one, inspite of its good performance, sacrifice the unbiasedness (this is preserved for the Henderson's one) of the variance components to guarantee their nonnegativity, and are not explicitly calculated.

Tests of hypothesis presented here are based on the Henderson's estimators. A numerical example for such a tests can be found at Sahai and Ojeda Sahai(2005).

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