

MATRICES THAT DEFINE SERIES OF PYTHAGOREAN TRIPLES THAT HAVE A TRIANGLE WITH ONE IRRATIONAL SIDE AS LIMIT

Martin William Bredenkamp

Assistant Prof. DSc, MSc, BSc Hons, BSc.,
Asia-Pacific International University, Thailand

Abstract

Making use of the universal set of Pythagorean triples, series of triples are defined where triple $n + 1$ is obtained by multiplying triple n with a specific 3×3 matrix. In terms of Pythagorean triangles, the shape of the limiting triangle in these series is a triangle with one of its sides having an irrational ratio with respect to the other sides. These specific matrices may be directly associated with the square roots of uneven positive integers (that are not perfect squares), and also some of the even positive integers since the limit of the powers of these matrices, applied to any 3×1 matrix of real numbers leads to a specific right-angled triangle that contains that square root as one of its sides.

Keywords: Series of Pythagorean triples; 3×3 matrix operators; square roots as limits

Introduction

All relatively prime Pythagorean triples have been defined (Bredenkamp-1, 2013) by indices i and j where i is an uneven positive integer and j is an even positive integer, i and j are relatively prime, and the three sides of the triangle are defined as follows:

$$\begin{aligned} u &= i^2 + ij \\ e &= j^2/2 + ij \\ h &= i^2 + ij + j^2/2 \end{aligned}$$

where u and e are the uneven- and even-numbered legs of the primitive right-angled triangle respectively, and h represents the hypotenuse.

Using this two-dimensional matrix of triangles as a universal set, subsets may be found where, if the triangles are arranged from smallest to greatest, a series of triangles is defined where the limit of the infinite series is a right-angled triangle that has one side irrational with respect to the other two sides (Bredenkamp-2, 2013). The triangles in these series occur with geometric regularity (Bredenkamp-3, 2013), and therefore formulae have been developed that generate the next member of a series.

Berggren (1934) and Price (2008) have shown that there are 3×3 matrices that may be used to generate one Pythagorean triple from another. In this paper we will show how that the formulae developed for the series (Bredenkamp-2, 2013) may indeed be expressed as these 3×3 matrices, and that these matrices, raised to infinite power, produce the triangles with one side being irrational.

The 45° Triangle

The subset of triangles that describes the series of the 45° triangle is A:

$$A = \{(i, j) | u = i^2 + ij, e = j^2/2 + ij, h^2 = u^2 + e^2, \text{ and } |e - u| = 1, \text{ where } (i + 1)/2 \text{ and } j/2 \in \mathbb{N}\}$$

Arranging these triangles in a series that has the component numbers, u, e, h, i and j increasing as the series progresses, enables the next member of the series to be described by an algebraic formula:

For the series $(i, j)_n$: $(i, j)_{n+1} = (i_n + j_n, 2i_n + j_n)$ (1)

Using the definition of u, e and h in terms of i and j , finding the next member of the series may be described as follows:

$$\begin{aligned} u_{n+1} &= 2h_n + 2e_n + u_n \\ e_{n+1} &= 2h_n + e_n + 2u_n \quad \text{and} \\ h_{n+1} &= 3h_n + 2e_n + 2u_n \end{aligned}$$

This may, indeed, be reformulated as a 3×3 matrix, retaining the order of (u, e, h) as ordered triplets in the matrix multiplication, and using the (3, 4, 5)-triangle as the first member of the series, which it is, the second triangle in the series is obtained (21, 20, 29):

$$(3 \quad 4 \quad 5) \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 21 \\ 20 \\ 29 \end{pmatrix}$$

Applying the matrix to the second triangle produces the third (119, 120, 169):

$$(21 \quad 20 \quad 29) \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 119 \\ 120 \\ 169 \end{pmatrix}$$

Therefore the third member of the series may be obtained by the square of the 3×3 matrix:

$$(3 \quad 4 \quad 5) \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}^2 = (3 \quad 4 \quad 5) \begin{pmatrix} 9 & 8 & 12 \\ 8 & 9 & 12 \\ 12 & 12 & 17 \end{pmatrix} = \begin{pmatrix} 119 \\ 120 \\ 169 \end{pmatrix}$$

We can therefore formulate a generalized equation for procuring series member n as follows:

$$(u_n \quad e_n \quad h_n) = (3 \quad 4 \quad 5) \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}^{n-1}$$

From there we have (Bredenkamp-2, 2013):

$$\lim_{n \rightarrow \infty} \left(\frac{h_n}{u_n} \right) = \sqrt{2} \quad \text{and} \quad \lim_{n \rightarrow \infty} \left(\frac{h_n}{e_n} \right) = \sqrt{2}$$

The 30/60° Triangle

For the 30/60° triangle a similar course may be pursued: The subset of triangles that describes the series of the 30/60° triangle is B:

$$B = \{(i, j) \mid u = i^2 + ij, \quad e = j^2/2 + ij, \quad h^2 = u^2 + e^2, \quad \text{and} \quad |h - 2u| = 1, \quad \text{where} \quad (i + 1)/2 \text{ and } j/2 \in \mathbb{N}\}$$

Arranging these triangles in a series as before, enables the next member of the series to be described as follows:

For the series $(i, j)_n$: $(i, j)_{n+1} = (i_n + j_n, 2i_n + 3j_n)$ (2)

Finding the next member of the series may be described as follows:

$$\begin{aligned} u_{n+1} &= 4h_n + 4e_n - u_n \\ e_{n+1} &= 8h_n + 7e_n - 4u_n \quad \text{and} \\ h_{n+1} &= 9h_n + 8e_n - 4u_n \end{aligned}$$

This may also be reformulated as a 3×3 matrix, retaining the order of (u, e, h) as ordered triplets in the matrix multiplication, and starting with the (3, 4, 5)-triangle ($n = 1$) the second triangle in the series is obtained (33, 56, 65):

$$(3 \quad 4 \quad 5) \begin{pmatrix} -1 & -4 & -4 \\ 4 & 7 & 8 \\ 4 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 33 \\ 56 \\ 65 \end{pmatrix}$$

As before, we can formulate a generalized equation for procuring series member n as follows:

$$(u_n \ e_n \ h_n) = (3 \ 4 \ 5) \begin{pmatrix} -1 & -4 & -4 \\ 4 & 7 & 8 \\ 4 & 8 & 9 \end{pmatrix}^{n-1}$$

From there we have (Bredenkamp-2, 2013):

$$\lim_{n \rightarrow \infty} \left(\frac{e_n}{u_n} \right) = \sqrt{3}$$

An alternative approach to triangles with the 30/60° triangle as limit is as follows (Bredenkamp-2, 2013):

$$(u_n \ e_n \ h_n) = (15 \ 8 \ 17) \begin{pmatrix} 7 & 4 & 8 \\ -4 & -1 & -4 \\ 8 & 4 & 9 \end{pmatrix}^{n-1}$$

with

$$\lim_{n \rightarrow \infty} \left(\frac{u_n}{e_n} \right) = \sqrt{3}$$

Other Series with Square Roots of Uneven Numbers as their Limits

Using as limit-triangles the right-angled triangles in the table below, which have the square root of any uneven number as the irrational leg, together with a rational other leg and a rational hypotenuse, new matrices are developed, together with series of rational numbers that have the respective irrational numbers as their limits:

hypotenuse e	2	3	4	5	6	7	8	9	10	11	12	$\frac{1}{3}$	14	15	16	17
rational leg	1	2	3	4	5	6	7	8	9	10	11	$\frac{1}{2}$	13	14	15	16
irrational leg	$\sqrt{3}$	$\sqrt{5}$	$\sqrt{7}$	3	$\sqrt{11}$	$\sqrt{13}$	$\sqrt{15}$	$\sqrt{17}$	$\sqrt{19}$	$\sqrt{21}$	$\sqrt{23}$	5	$\sqrt{27}$	$\sqrt{29}$	$\sqrt{31}$	$\sqrt{33}$

Here follows the starting triangles, matrices, and limit formulae of the first few of these series, beginning with $\sqrt{5}$. Note that there are two sets of data for every number. It is also interesting to compare the matrices of these alternate sets:

root	Initial triplet	matrix	Limit formula
$\sqrt{5}$	(3 4 5)	$\begin{pmatrix} -9 & -8 & -12 \\ 8 & 9 & 12 \\ 12 & 12 & 17 \end{pmatrix}$	$\lim_{n \rightarrow \infty} \left(\frac{2e_n}{u_n} \right) = \sqrt{5}$
	(3 4 5)	$\begin{pmatrix} 9 & 8 & 12 \\ -8 & -9 & -12 \\ 12 & 12 & 17 \end{pmatrix}$	$\lim_{n \rightarrow \infty} \left(\frac{2u_n}{e_n} \right) = \sqrt{5}$
$\sqrt{7}$	(3 4 5)	$\begin{pmatrix} -161 & -144 & -216 \\ 144 & 127 & 192 \\ 216 & 192 & 289 \end{pmatrix}$	$\lim_{n \rightarrow \infty} \left(\frac{3e_n}{u_n} \right) = \sqrt{7}$
	(3 4 5)	$\begin{pmatrix} 127 & 144 & 192 \\ -144 & -161 & -216 \\ 192 & 216 & 289 \end{pmatrix}$	$\lim_{n \rightarrow \infty} \left(\frac{3u_n}{e_n} \right) = \sqrt{7}$
$\sqrt{11}$	(15 8 17)	$\begin{pmatrix} -449 & -300 & -540 \\ 300 & 199 & 360 \\ 540 & 360 & 649 \end{pmatrix}$	$\lim_{n \rightarrow \infty} \left(\frac{5e_n}{u_n} \right) = \sqrt{11}$
	(3 4 5)	$\begin{pmatrix} 199 & 300 & 360 \\ -300 & -449 & -540 \\ 360 & 540 & 649 \end{pmatrix}$	$\lim_{n \rightarrow \infty} \left(\frac{5u_n}{e_n} \right) = \sqrt{11}$

Clearly, the complementary matrices for each root have their first and second rows interchanged, followed by their first and second columns. This is also reflected in the limit formula where the legs are interchanged (first and second rows/columns represent the legs).

Further Applications of the Matrices

The choice of the initial triangle for a series is in a sense arbitrary. A limit is set on the differences of the operational outcomes of the sides that are involved with the limit, which is less than or equal to one. When that limit is increased, more triangles qualify as the starting points of series of triangles, and as the triangles increase in size in the series, all these series have the same triangle as their limit. Even with 1 as the limit, some square roots have several series (Bredenkamp-2, 2013). As an example, using the 45° matrix, but beginning with the (5, 12, 13)-triangle, a series develops that still has the 45° triangle as its limit:

$$(5 \quad 12 \quad 13) \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 55 \\ 48 \\ 73 \end{pmatrix}$$

The series then is:

{(5, 12, 13); (55, 48, 73); (297, 304, 425); (1755, 1748, 2477); (10205, 10212, 14437);}

Notice the difference between the legs is 7 throughout, which means as the triangles increase in size, so the difference of 7 between the legs becomes less significant, and the triangle approaches the shape of the 45° triangle.

It turns out that any numbers may be used as the initial 3 × 1 matrix, and the limit still is the 45° triangle. If negative numbers are incorporated in the matrix, the series of triangles may become triangles with negative sides, but the proportions of these negative sides still have the 45° triangle as the limit. Consider the following example, and notice that the difference in the “legs” remains 12 throughout:

{(-10, 2, -4); (-14, -26, -28); (-122, -110, -164); (-670, -682, -956); (-3946, -3934, -5572);}

Even irrational numbers may be used:

{(√2, π, √7); (12.9889, 11.2615, 17.0489); (69.6097, 71.3371, 99.6474); (408.1239, 409.8513, 580.8358); (2389.4982, 2387.7708, 3378.4579);}

Even here the “legs” remain within less than two from each other, and as the numbers increase, that difference becomes insignificant.

Conclusion

Matrices that Define Square Roots.

For the series of triangles that have as their limit the 45° triangle, one of the three Berggren matrices (Berggren, 1934),

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

is used to propagate the series, the initial triangle being the (3, 4, 5) triangle. It so happens that any real number may be used in the three slots of the initial 3 × 1 matrix [in the place of the triple for the (3, 4, 5) triangle], and the limit still becomes the 45° triangle (sometimes with negative sides). The ratio of the largest number to either of the smaller numbers has as its limit √2. The Berggren 3 × 3 matrix may therefore be associated with √2 and may even be called the √2 matrix.

In the same way, every uneven number that is not a perfect square has a matrix associated with its root. Some of the even numbers, by similar procedures, also have matrices associated with their roots, since by using other triangles these square roots are implicated (Bredenkamp-2, 2013). There is therefore an infinite number of 3 × 3 matrices that relate Pythagorean triples with each other, but these matrices may also be used to describe a series of 1 × 3 matrices that have as their limits ratios that describe the same irrational numbers, irrespective of the initial real numbered 1 × 3 matrix.

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