EXTENSION OF HOMOTOPY PERTURBATION METHOD FOR SOLVING NONLINEAR SYSTEMS

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Abstract

In this paper, a homotopy perturbation method (HPM) is extended and applied for solving system of nonlinear equations of *n*-dimensional with *n*-variables. Also, numerical examples are used to show the performance of the presented method, on a series of examples published in the literature, and to compare with other literature methods.

Keywords: Homotopy method, perturbation method, System of nonlinear equations, Iterative method, Newton's method

Introduction

Homotopy perturbation methods HPM play a very important role in solving several mathematical problems such as linear and nonlinear system equations, differential equation and integral equations (He, 2000-2006; Soltanian, 2010; Dehghan, 2008-2011). The basic idea of HPM is to simplify the difficult equation systems by converting them into either linear or nonlinear system equations so that they can be solved. In the recent years, HPM attracts the attention of the authors, because solutions of this method offer a high degree of accuracy and convergence (El-Shahed, 2005; Ghasemi, 2006; Javidi, 2007; Abbasbandy2003).

H. He (He; 2005) suggested an iterative method for solving the nonlinear equations by rewriting the given nonlinear equation as a system of coupled equations. This technique has been used by Chun (Chun2005) and Noor et. al. (Noor et.al,2006-2007) to suggest some higher order convergent iterative methods for solving nonlinear equations. In 2007, A. Golbabai et.al(Golbabai et.al,2007) applied HPM for solving system of nonlinear equations in two dimensions by expanding the variables into Taylor series. In this research we extend this method to solve system of nonlinear equations of *n*-dimension with *n*-variables. For validating our proposed method, we compare its result with the Newton's Raphson method (Abbasbandy, 2003).

Extended homotopy perturbation method

Suppose we have a system of nonlinear equations of the following form

$$F(X)=\begin{cases} f_1(x_1,x_2,\ldots,x_n)=0 \\ f_2(x_1,x_2,\ldots,x_n)=0 \\ \vdots \\ f_n(x_1,x_2,\ldots,x_n)=0 \end{cases}, \quad X=(x_1,x_2,\ldots,x_n)\in R^n \tag{1}$$

where $f_i:\mathbb{R}^n\to\mathbb{R}$ and $F:\mathbb{R}^n\to\mathbb{R}^n$, and the functions f_i is differentiable up to any desired order (Burden,2001). To illustrate the basic ideas of homotopy perturbation method, we construct a homotopy $\mathbb{R}^n\times[0,1]\to\mathbb{R}$ which satisfies

$$H(\overline{X},p)=pF(\overline{X})+(1-p)[F(\overline{X})-F(X_0)]=0, \quad \overline{X}\in\mathbb{R}^n, \quad p\in[0,1], \tag{2}$$

here

$$H(\overline{X},p)=\begin{cases} H_1(\overline{X},p)=f_1(\overline{X})-f_1(X_0)+pf_1(X_0)=0, \\ H_2(\overline{X},p)=f_2(\overline{X})-f_2(X_0)+pf_2(X_0)=0, \\ \vdots \\ H_n(\overline{X},p)=f_n(\overline{X})-f_n(X_0)+pf_n(X_0)=0, \end{cases} \quad ,\overline{X}\in\mathbb{R}^n, p\in[0,1] \tag{3}$$

where p is embedding parameter, $X_0=(x_1^{(0)},x_2^{(0)},\ldots,x_n^{(0)})$ is an initial approximation of *Eq*. (1).

It is obvious that

$$H(\overline{X},0)=F(\overline{X})-F(X_0)=0, \quad H(\overline{X},1)=F(\overline{X})=0. \tag{4}$$

The embedding parameter p monotonically increases from zero to unit as a trivial problem $H(\overline{X},0)=F(\overline{X})-F(\overline{X}_0)=0$ is continuously deformed to original problem $H(\overline{X},1)=F(\overline{X})=0$. The (HPM) uses the homotopy parameter p as an expanding parameter to obtain:

$$\overline{X} = \begin{cases} \overline{x}_1 = x_1^{(0)} + p x_1^{(1)} + p^2 x_1^{(2)} + \dots, \\ \overline{x}_2 = x_2^{(0)} + p x_2^{(1)} + p^2 x_2^{(2)} + \dots, \\ \vdots \\ \overline{x}_n = x_n^{(0)} + p x_n^{(1)} + p^2 x_n^{(2)} + \dots \end{cases} \tag{5}$$

The approximate solution of *Eq.* (1), therefore, can be readily obtained:

$$X = \lim_{p\to 1} \overline{X} = \begin{cases} \lim\limits_{p\to 1} \overline{x}_1 = x_1^{(0)} + x_1^{(1)} + x_1^{(2)} + \dots, \\ \lim\limits_{p\to 1} \overline{x}_2 = x_2^{(0)} + x_2^{(1)} + x_2^{(2)} + \dots, \\ \vdots \\ \lim\limits_{p\to 1} \overline{x}_n = x_n^{(0)} + x_n^{(1)} + x_n^{(2)} + \dots \end{cases} \tag{6}$$

The convergence of the series in (6) has been studied and discussed in (He,1999).

For the application of (HPM) to Eq. (1) we can rewrite (3) by expanding $f_i(\overline{X})$ into a Taylor series around X_0 as follows:

$$\begin{gathered} \left[f_k(X_0) + \frac{1}{1!}[\sum_{i=1}^{n}(\overline{x}_i - x_i^{(0)}) f_{k,i}(X_0)] + \frac{1}{2!}[\sum_{i=1}^{n}\sum_{j=1}^{n}(\overline{x}_i - x_i^{(0)})(\overline{x}_j - x_j^{(0)}) f_{k,ij}(X_0)] + \dots \right] - \\ - f_k(X_0) + p f_k(X_0) = 0 \\ k = 1, 2, \dots, n \end{gathered} \tag{7}$$

where $f_{1,i} = \dfrac{\partial f_1}{\partial x_1}$, and $f_{1,ij} = \dfrac{\partial^2 f_1}{\partial x_i \partial x_j}$. Substitution of (5) into (7) yields

$$\begin{gathered} \left[f_k(X_0) + \frac{1}{1!}[\sum_{i=1}^{n}(\sum_{m=0}^{\infty} p^m x_i^{(m)} - x_i^{(0)}) f_{k,i}(X_0)] + \right. \\ \left. \frac{1}{2!}[\sum_{i=1}^{n}\sum_{j=1}^{n}(\sum_{m=0}^{\infty} p^m x_i^{(m)} - x_i^{(0)})(\sum_{m=0}^{\infty} p^m x_j^{(m)} - x_j^{(0)}) f_{k,ij}(X_0)] + \dots \right] \\ - f_k(X_0) + p f_k(X_0) = 0, \quad k = 1, 2, \dots, n \end{gathered} \tag{8}$$

By equating the terms with identical powers of p, we have

$$p^{(0)}:\begin{cases} f_1(X_0)-f_1(X_0)=0 \\ f_2(X_0)-f_2(X_0)=0 \\ \quad\vdots \\ f_n(X_0)-f_n(X_0)=0 \end{cases} \tag{9}$$

$$p^{(1)}:\begin{cases} \sum_{i=1}^{n} x_i^{(1)} f_{1,i}(X_0)+f_1(X_0)=0 \\ \sum_{i=1}^{n} x_i^{(1)} f_{2,i}(X_0)+f_2(X_0)=0 \\ \quad\vdots \\ \sum_{i=1}^{n} x_i^{(1)} f_{n,i}(X_0)+f_n(X_0)=0 \end{cases} \tag{10}$$

$$p^{(2)}:\begin{cases} \sum_{i=1}^{n} x_i^{(2)} f_{1,i}(X_0)+\frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^{(1)} x_j^{(1)} f_{1,ij}(X_0)=0 \\ \sum_{i=1}^{n} x_i^{(2)} f_{2,i}(X_0)+\frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^{(1)} x_j^{(1)} f_{2,ij}(X_0)=0 \\ \quad\vdots \\ \sum_{i=1}^{n} x_i^{(2)} f_{n,i}(X_0)+\frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^{(1)} x_j^{(1)} f_{n,ij}(X_0)=0 \end{cases} \tag{11}$$

From Eq. (9–11), the sequences $x_i^{(k)}$ can be obtained in the following form

$$\begin{bmatrix} x_1^{(1)} \\ \dots \\ x_n^{(1)} \end{bmatrix} = -\begin{bmatrix} f_{1,1}(X_0) & \dots & f_{1,n}(X_0) \\ \dots & \dots & \dots \\ f_{n,1}(X_0) & \dots & f_{n,n}(X_0) \end{bmatrix}^{-1} \begin{bmatrix} f_1(X_0) \\ \dots \\ f_n(X_0) \end{bmatrix} \tag{12}$$

$$\begin{bmatrix} x_1^{(2)} \\ \dots \\ x_n^{(2)} \end{bmatrix} = -\begin{bmatrix} f_{1,1}(X_0) & \dots & f_{1,n}(X_0) \\ \dots & \dots & \dots \\ f_{n,1}(X_0) & \dots & f_{n,n}(X_0) \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^{(1)} x_j^{(1)} f_{1,ij}(X_0) \\ \dots \\ \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^{(1)} x_j^{(1)} f_{n,ij}(X_0) \end{bmatrix} \tag{13}$$

Substituting all the above terms into (6), we can obtain the zero of Eq. (1) as follows:

$$X = \begin{bmatrix} x_1^{(0)} \\ \dots \\ x_n^{(0)} \end{bmatrix} - \begin{bmatrix} f_{1,1}(X_0) & \dots & f_{1,n}(X_0) \\ \dots & \dots & \dots \\ f_{n,1}(X_0) & \dots & f_{n,n}(X_0) \end{bmatrix}^{-1} \begin{bmatrix} f_1(X_0) \\ \dots \\ f_n(X_0) \end{bmatrix} -$$

$$\begin{bmatrix} f_{1,1}(X_0) & \dots & f_{1,n}(X_0) \\ \dots & \dots & \dots \\ f_{n,1}(X_0) & \dots & f_{n,n}(X_0) \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^{(1)} x_j^{(1)} f_{1,ij}(X_0) \\ \dots \\ \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i^{(1)} x_j^{(1)} f_{n,ij}(X_0) \end{bmatrix} + \dots \qquad (14)$$

This formula allows us to suggest the following iterative methods for solving nonlinear system equations. (1).

Algorithms

In this section, we presented two algorithms for solving system of nonlinear equations:

Algorithm.1.

For a given $Z^{(k)} = [z_1^{(k)}, z_2^{(k)}, \dots, z_n^{(k)}]^T$ calculate the approximation solution $Z^{(k+1)} = [z_1^{(k+1)}, z_2^{(k+1)}, \dots, z_n^{(k+1)}]^T$ for k =1, 2, … by the iterative scheme

$$\begin{bmatrix} z_1^{(k+1)} \\ \dots \\ z_n^{(k+1)} \end{bmatrix} = \begin{bmatrix} z_1^{(k)} \\ \dots \\ z_n^{(k)} \end{bmatrix} - \begin{bmatrix} f_{1,1}(Z^{(k)}) & \dots & f_{1,n}(Z^{(k)}) \\ \dots & \dots & \dots \\ f_{n,1}(Z^{(k)}) & \dots & f_{n,n}(Z^{(k)}) \end{bmatrix}^{-1} \begin{bmatrix} f_1(Z^{(k)}) \\ \dots \\ f_n(Z^{(k)}) \end{bmatrix}, \qquad (15)$$

which is the Newton–Raphson method for n dimension and we will denote it by (NM).

Algorithm.2.

For a given $Z^{(k)} = [z_1^{(k)}, z_2^{(k)}, \dots, z_n^{(k)}]^T$ calculate the approximation solution $Z^{(k+1)} = [z_1^{(k+1)}, z_2^{(k+1)}, \dots, z_n^{(k+1)}]^T$ for k =0, 1, 2, … by the iterative scheme

$$\begin{bmatrix} L_1^{(k)} \\ \dots \\ L_n^{(k)} \end{bmatrix} = - \begin{bmatrix} f_{1,1}(Z^{(k)}) & \dots & f_{1,n}(Z^{(k)}) \\ \dots & \dots & \dots \\ f_{n,1}(Z^{(k)}) & \dots & f_{n,n}(Z^{(k)}) \end{bmatrix}^{-1} \begin{bmatrix} f_1(Z^{(k)}) \\ \dots \\ f_n(Z^{(k)}) \end{bmatrix} \qquad (16)$$

$$\begin{bmatrix} z_1^{(k+1)} \\ \dots \\ z_n^{(k+1)} \end{bmatrix} = \begin{bmatrix} z_1^{(k)} \\ \dots \\ z_n^{(k)} \end{bmatrix} + \begin{bmatrix} L_1^{(k)} \\ \dots \\ L_n^{(k)} \end{bmatrix} - \begin{bmatrix} f_{1,1}(Z^{(k)}) & \dots & f_{1,n}(Z^{(k)}) \\ \dots & \dots & \dots \\ f_{n,1}(Z^{(k)}) & \dots & f_{n,n}(Z^{(k)}) \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} L_i^{(1)} L_j^{(1)} f_{1,ij}(Z^{(k)}) \\ \dots \\ \frac{1}{2!}\sum_{i=1}^{n}\sum_{j=1}^{n} L_i^{(1)} L_j^{(1)} f_{n,ij}(Z^{(k)}) \end{bmatrix} \quad (17)$$

and we will denote it by (HPM). We also remark that if $f_{k,ij} = 0, \forall k, i, j = 1, 2, \dots, n$, then Algorithm 2 reduces to the Newton Method, that is, Algorithm 1.

Applications

We present some examples to illustrate the efficiency of our proposed method. We apply the algorithm 2. and compare the results with the standard Newton–Raphson method NM. We use the following stopping criteria for computer programs, $|x_{n+1} - x_n| < \varepsilon$ or $|f(x_n)| < \varepsilon$, and $\varepsilon = 10^{-15}$. Here, the algorithm is performed by Maple 15 with 20 significant digits, but only 15 digits are displayed.

In Tables 1-7 we list the results obtained by extended homotopy perturbation method HPM and compare it with the Newton–Raphson method (NM) using several values of the initial guess approximation x_0. As we see from these Tables, it is clear that the result obtained by the present method is very superior to that obtained by the other method.

Small systems of nonlinear equations.

Example 1.

In a case of one dimension, consider the following nonlinear functions (Noor et.al,2010), $f_1(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5$, with $x_0 = -1$ and $f_2(x) = e^{x^2+7x-30} - 1$ with $x_0 = 3.5$

Example 2.

In a case two dimensions, consider the following systems of nonlinear functions (Golbabai et.al,2007)

$$F_3(X) = \begin{cases} f_1(x,y) = x^2 - 10x + y^2 + 8 = 0 \\ f_2(x,y) = xy^2 + x - 10y + 8 = 0 \end{cases}, \quad (x_0, y_0) = (0.8, 0.8).$$

$$F_4(X) = \begin{cases} f_1(x,y) = x^4 y - xy + 2x - y - 1 = 0 \\ f_2(x,y) = ye^{-x} + x - y - e^{-1} = 0 \end{cases}, \quad (x_0, y_0) = (0.8, 0.8).$$

Table 1. Numerical results for Example 2 (F_3)

F_3	NM		HPM	
NI	x	y	x	y
1	**.987766198459447**	**.985772541912097**	**0.998270352547736**	**0.997314611742872**
2	**.999935603124850**	**.999916479630138**	**0.999999995819455**	**0.999999993343152**
3	**.999999997926065**	**.999999997265203**	**0.999999999999999**	**1.000000000000000**

Table 2. Numerical results for Example 2 (F_4)

F_4	NM		HPM	
NI	x	y	x	Y
1	**1.202154745465234**	**1.252500021237429**	**0.530135422051184**	**0.375112183418822**
2	**1.066458500832922**	**1.071778294601873**	**1.167578846210930**	**0.826948359050393**
3	**1.008970459700100**	**1.009951147648418**	**1.002859536677230**	**0.999714844009216**
4	**1.000190149835321**	**1.000217129214667**	**1.000000085472060**	**1.000000076036040**

Example 3.

In a case three dimensions, consider the following systems of nonlinear functions (Hosseini et.all,2010;Faid,2006;Vahididi et.al,2012).

$$F_5(X)=\begin{cases} f_1(x,y,z)=15x+y^2-4z-13=0 \\ f_2(x,y,z)=x^2+10y-e^{-z}-11=0, \\ f_3(x,y,z)=y^3-25z+22=0 \end{cases} \qquad (x_0, y_0, z_0)=(5, 4, 2).$$

Table 3. Numerical results for Example 3 (F_5)

F_5	Newton-Raphson			HPM		
NI	X	Y	z	x	y	z
4	**1.04218011628512**	**1.03108984243725**	**0.923740415181686**	**1.04214956379484**	**1.03109124240577**	**0.923848150437484**
5	**1.04214956059142**	**1.03109127170003**	**0.923848154861335**	**1.04214956057693**	**1.03109127183940**	**0.923848154879367**
6	**1.04214956057693**	**1.03109127183940**	**0.923848154879367**	**1.04214956057693**	**1.03109127183940**	**0.923848154879367**
7	**1.04214956057693**	**1.03109127183940**	**0.923848154879367**			

$$F_6(X)=\begin{cases} f_1(x,y,z)=3x-\cos(yz)-0.5=0 \\ f_2(x,y,z)=x^2-81(y+0.1)^2+\sin z+1.06=0 \\ f_3(x,y,z)=e^{-xy}+20z+\dfrac{10\pi-3}{3}=0 \end{cases}, \qquad (x_0, y_0, z_0)=(5, 5, 2).$$

Table 4. Numerical results for Example 3. (F_6)

F_6	Newton-Raphson			HPM		
NI	X	Y	z	x	y	z
9	**0.496575184963897**	**-0.261732255793898**	**-0.530372756344176**	**0.498144684589491**	**0.498144684589491**	**0.498144684589491**
10	**0.498036630557039**	**-0.211523651685691**	**-0.529140611531331**	**0.498144684589491**	**-0.199605895543779**	**-0.528825977573387**
11	**0.498138890356978**	**-0.200241563678410**	**-0.528842599308139**			
12	**0.498144666210892**	**-0.199607907114003**	**-0.528826030181812**			
13	**0.498144684589305**	**0.498144684589305**	**-0.528825977573917**			
14	**0.498144684589491**	**-0.199605895543779**	**-0.528825977573387**			
15	**0.498144684589491**	**-0.199605895543779**	**-0.528825977573387**			

Table 5. Numerical results for Examples 1-3

$F(X)$	f_1	f_2	F_3	F_4	F_5	F_6
$\varepsilon = 10^{-15}$	Number of iterations					
NM	8	12	5	8	7	15
HPM	5	8	4	6	6	10

Large systems of nonlinear equations

In this subsection, we test HPM with some sparse systems and with *n* unknown variables. In examples 4 to 6, we compare the NR method with the proposed method HPM focusing on iteration numbers. In the previous studies, they used $\|F(x^{(n)})\|_2 < 10^{-13}$ as stop criterion.

Example 4.

Consider the following system of nonlinear equations [31]:

$$F_7 : f_i = e^{x_i} - 1, \quad i = 1, 2, ..., n.$$

The exact solution of this system is $x^* = [0,0,...,0]^T$. To solve this system, we set $X = [0.5, 0.5, ..., 0.5]^T$ as an initial value. Table 6 has shown this result.

Example 5.

Consider the following system of nonlinear equations (Hafiz and Bahgat,2012)

$$F_8 : f_i = x_i^2 - \cos(x_i - 1), \quad i = 1, 2, ..., n.$$

One of the exact solutions of this system is $x^* = [1,1,...,1]^T$. To solve this system, we set $x_0 = [2,2.,...,2]^T$ as an initial value. The results are presented in Table 6.

Example 6.

Consider the following system of nonlinear equations (Hafiz and Bahgat,2012):

$$F_9 : f_i = \cos x_i - 1, \quad i = 1, 2, ..., n.$$

One of the exact solutions of this system is $x^* = [0,0,...,0]^T$. To solve this system, we set

$x_0 = [2,2.,...,2]^T$ as an initial guess. The results are presented in Table 6.

Table 6. Numerical results for Example 4, 5 and 6.

Number of iterations	F_7	F_8	F_9	F_7	F_8	F_9	F_7	F_8	F_9
$\varepsilon = 10^{-13}$	n=50			n=75			n=100		
NR	5	7	21	5	7	21	5	7	21
HPM	4	5	15	4	5	15	4	5	15

Table 7. Numerical results for Examples 1-6 for several values of the initial guess

f_1	x_0	-2.5	-2.3	-2.1	-2	-1.8	-1
	NM	11	10	9	8	7	6
	HPM	7	6	6	5	5	4

f_2	x_0	3.1	3.2	3.3	3.4	3.5	3.6
	NM	6	8	9	10	12	13
	HPM	4	5	6	7	8	9
F_3	X_0	(-1.5,-1)	(0,0)	(.5,.5)	(.8,.8)	(1.5,0)	(0,1.5)
	NM	7	6	6	5	6	6
	HPM	6	5	4	4	4	4
F_4	X_0	(.6,.4)	(.5,.5)	(.5,.7)	(.9,.9)	(1.2,1.2)	(1.5,0)
	NM	7	7	7	9	6	7
	HPM	6	6	5	4	4	5
F_5	X_0	(3,4,4)	(5,4,2)	(6,4,3)	(6,3,1)	(7,2,1)	(8,1,1)
	NM	7	7	8	8	Divergent	21
	HPM	5	6	6	5	5	5
F_6	X_0	(4,4,2)	(5,5,2)	(6,4,1)	(5,2,0)	(5,1,0)	(4,0,1)
	NM	11	15	11	10	9	9
	HPM	8	10	8	7	6	8
F_7,	X_0	(.1,.1,...,.1)	(.3,.3,...,.3)	(.5,.5,...,.5)	(1,1,...,1)	(1.5,1.5,...,1.5)	(2,2,...,2)
n=50	NM	4	4	5	6	6	7
	HPM	3	3	4	4	5	6
F_8	X_0	(0,0,...,0)	(.2,.2,...,.2)	(.5,.5,...,.5)	(1.1,1.1,...,1.1)	(1.5,1.5,...,1.5)	(2,2,...,2)
n=50	NM	6	8	6	4	5	6
	HPM	6	7	5	3	4	5
F_9	X_0	(.1,.1,...,.1)	(.2,.2,...,.2)	(.5,.5,...,.5)	(1,1,...,1)	(1.5,1.5,...,1.5)	(2,2,...,2)
n=50	NM	18	19	21	21	22	21
	HPM	13	14	15	15	15	15

Conclusions

The Homotopy perturbation method was extended and applied to the numerical solution for solving system of nonlinear equations. The numerical examples show that our method is very effective and efficient. Moreover, our proposed method provides highly accurate results in a less number of iterations as compared with the Newton–Raphson method, when the initial value x_0 is well chosen. It is an open problem to determine the most appropriate choice of the initial guess.

Recommendation In future works i will prove the extended homotopy method.

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