RHEOLOGICAL RESPONSES OF VISCOELASTIC MODELS UNDER DYNAMICAL LOADING

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Abstract

The purpose of this paper is to study the rheological responses of four and five-parameter viscoelastic models under dynamic loading. These models are chosen for studying elastic, viscous, and retarded elastic responses to shearing stress. The viscoelastic specimen is chosen which closely approximates the actual behavior of a polymer. The module of elasticity and viscosity coefficients are assumed to be space dependent i.e. functions of $'x'$ in non-homogeneous case and stress-strain are harmonic functions of time $'t'$. The complex viscosity of five parameter model is calculated. The expression for relaxation time for five parameters and four parameter viscoelastic models have been obtained by using a constitutive equation. The dispersion equation is obtained by using Ray techniques. The rheological responses for both models under dynamical loading are shown graphically. Also, the five parameter model is justified with the help of cyclic loading for maxima or minima.

Keywords: Shear Waves, Viscoelastic Media, Asymptotic Method, Dynamic Loading, Friedlander Series

Introduction

The viscoelasticity theory is used in the field of solid mechanics, seismology, exploration geophysics, acoustics and engineering. The solutions of many problems related to wave-propagation in homogeneous media are available in many literatures of continuum mechanics of solids. However, in the recent years, the interest has arisen to solve the problems connected with non-homogeneous bodies. These problems are useful to understand the properties of polymeric materials and industrial related applications. The practice and theoretic analysis indicate that the dynamic loadings (such as earthquake, tsunami, raging billow and vibration) are the important factor and mainly the power of inducing geological disaster of soft rock-soil. The rheological mechanic's response and rheological parameters of specimen are tested and analyzed under dynamic loading, the viscoelastic-plasticity rheological dynamic model is established, and new rheological equation is deduced. The vibrations in earthquakes are due to differences in dynamic characteristics therefore the cyclic stress-strain behavior of material play a vital role for reliable prediction of the seismic response. Many researchers studied structural pounding during earthquakes. The lack of information concerns multi-dimensional waves in viscoelastic-media, and in particular for non-homogeneous media, therefore, a formal study of non-homogeneous viscoelastic models under dynamic loading is presented.

Modeling and model parameter estimation is of great importance for a correct prediction of the foundation behavior. Many researchers Alfrey (1944), Barberan and Herrera (1966), Achenbach and Reddy (1967), Bhattacharya and Sengupta (1978), and Acharya *et al.* (2008) formulated and developed this theory. Further, Bert and Egle (1969), Abd-Alla and Ahmed (1996) , Batra (1998) successfully applied this theory to wave-propagation in homogeneous, elastic media. Murayama and Shibata (1961), Schiffman *et al.* (1964) have proposed higher order viscoelastic models of five and seven parameters to represent the soil behavior. Jankowski *et al.* (1998) discussed the linear viscoelastic model and the nonlinear viscoelastic model. Anagnostopoulos (1988) studied the linear viscoelastic model of collision to simulate structural pounding. Jankowski *et al*. (2005) studied the pounding of superstructure segments in bridges with the help of a linear viscoelastic model. Muthukumar and DesRoches (2006) did a comparison study using two single degree of freedom (SDOF) systems for capturing pounding. Westermo (1989) suggested linking buildings with beams which can transmit the forces between them eliminating dynamic contacts. Kakar *et al*.; (2012) and Kaur *et al*.; (2012) analyzed viscoelastic models under Dynamic Loading, Recently, Kakar and Kaur (2013) analyzed five parameter model of the propagation of cylindrical shear waves in non-homogeneous viscoelastic media.

The behavior of real materials cannot be completely represented by the simple Maxwell and Kelvin Model. More complicated models are required with a greater flexibility in portraying the response of actual material. Maxwell unit in parallel with a spring is the standard linear solid and a Maxwell unit in parallel with a dashpot is the viscous model. A four parameter model consisting of two springs and two despots may be regarded as a Maxwell unit in series with a Kelvin unit is capable of all the three basic viscoelastic patterns.

Thus it imparts instantaneous elastic response because of the free spring, viscous flow because of the free dashpot and finally delayed elastic response from the Kelvin unit. This study also focuses on five parameter model in which the module of elasticity and viscosity coefficients are assumed to be space dependent i.e. functions of $'x'$. Further, shearing strain $'a'$ and stress $'\sigma'$ are taken as harmonic functions of time $'t'$ i.e. $a = a_0 e^{i\omega t}$ and $\sigma = G_0 a = G_0 a_0 e^{i\omega t} = \sigma_0 e^{i\omega t}$. The model is justified with the help of cyclic loading for maxima or minima. Here all the characteristics of the viscoelastic properties of the material, as determined under the harmonic oscillations, are the frequency dependence of the components of the complex modulus or the frequency dependence of the phase angle.

Constitutive Relation For Five Parameter Model

The five parameter model consists of two springs $S_1(G_1), S_2(G_2)$ where G_1 and G_2 are the modulli of elasticity associated to them and three dash-pots $D_2(\eta_2), D_{2'}(\eta_{2'}), D_3(\eta_3)$ where $\eta_2, \eta_{2'}$ and η_3 are the Newtonian viscosity coefficients associates to these elements. The module of elasticity and viscosity coefficients are assumed to be space dependent i.e. functions of $'x'$ in inhomogeneous case taken into consideration. Unidirectional problem is formed by taking the material in the form of filament of non-homogeneous viscoelastic material by taking one end at x = 0. The co-ordinate x is measured positive in the direction of the axis of the filament. Time is specified by t, and σ, γ and u respectively specify the only non-zero components of stress, shearing strain and particle displacement. The model has be divided into three sections, I, II, III. In fig.1, the section I, section II and section III has one spring $S_1(G_1)$, two dash-pots $D_2(\eta_2), D_{2'}(\eta_{2'})$ one spring $S_2(G_2)$ and one dash-pot $D_3(\eta_3)$ respectively.

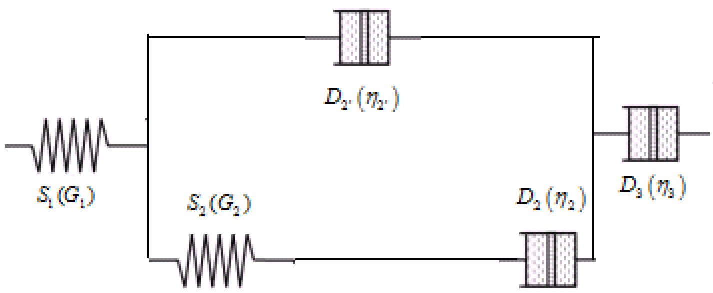

Figure-1 Five parameter viscoelastic model

Under the supper- supposition principle strains are added in the case of series connections and stresses are added when they are in parallel. Now if a_1, a_2, a_3 be the three shearing strains elongations in respective sections connected in series, then total elongation is $a = a_1 + a_2 + a_3$. The total stress in the network remains the same. In each section but in the case of section II which is sub-divided into two sections is added i.e. $\sigma = \sigma_1 + \sigma_2$, where σ_1 and σ_2 are the stresses in the sub-sections. Relation for stress and strain for $D_{2'}(\eta_{2'})$ for section II (represented by single dash-pot) is

$$\sigma_1 = \eta_{2'} a_2 \tag{1}$$

Since the sub-section II is represented by a Maxwell- element, then the relation is expressed as

$$\left(\frac{D}{G_2} + \frac{1}{\eta_2}\right)\sigma_2 = D(a_2) \tag{2}$$

Since, $\sigma = \sigma_1 + \sigma_2$ for Section II , therefore

$$\left(D + \frac{G_2}{\eta_2}\right)\sigma = \eta_{2'}\left[D^2 + G_2\left(\frac{1}{\eta_{2'}} + \frac{1}{\eta_2}\right)D\right]a_2 \tag{3}$$

For section I, for the spring $S_1(G_1)$, the stress-strain relation is given by

$$\sigma = G_1 a_1 \tag{4}$$

For Section III; for the dash-pot $D_3(\eta_3)$, the stress –strain relation is given by

$$\sigma = \eta_3 \dot{a}_3 \tag{5}$$

The Stress-strain relation for the model representing the viscoelastic body for total stress (σ) and strain (a) can be obtained from Eq. (3), Eq. (4) and Eq. (5) as:

$$\left[D^2 + \left\{\frac{G_1}{\eta_{2'}} + \frac{G_1}{\eta_3} + \frac{G_2}{\eta_{2'}} + \frac{G_2}{\eta_2}\right\}D + \left\{\frac{G_1}{\eta_{2'}}\frac{G_2}{\eta_2} + \frac{G_1}{\eta_3}\left(\frac{G_2}{\eta_{2'}} + \frac{G_2}{\eta_2}\right)\right\}\right]\sigma = G_1\left\{D^2 + \left(\frac{G_2}{\eta_{2'}} + \frac{G_2}{\eta_2}\right)D\right\}a \tag{6}$$

Now we take

$$\tau_{ij}^{-1} = \theta_{ij} = \frac{G_i}{\eta_j} = \frac{S_i(G_i)}{Dj(\eta_j)} \tag{7}$$

Where, $S_i(G_i)$ elastic modulus of spring and $Dj(\eta_j)$ = viscosity of dash-pot, $\tau_{ij} = \frac{\eta_j}{G_i}$, $(i = 1,2\ ;\ j = 2', 2, 3)$ Using, Eq. (6) and Eq. (7), we get

$$\left[D^2 + \left\{(\theta_{12'} + \theta_{13}) + (\theta_{22} + \theta_{22'})\right\}D + \left\{\begin{array}{l}\theta_{12'}.\theta_{22} \\ +\theta_{13}(\theta_{22} + \theta_{22'})\end{array}\right\}\right]\sigma = G_1\left\{D^2 + (\theta_{22} + \theta_{22'})D\right\}a \tag{8}$$

Put $R_1 = \theta_{12'} + \theta_{13}$, $R_2 = \theta_{22} + \theta_{22'}$, R3 = $R_3 = \theta_{12'}.\theta_{22}$, R4 = $R_4 = \theta_{13}$ in Eq. (8), we get

$$\left[D^2 + \left(R_1 + R_2\right)D + \left(R_3 + R_4\right)\right]\sigma = G_1\left\{D^2 + R_2 D\right\}a = \left(G_1 D^2 + G_1\ R_2 D\right)a \tag{9}$$

The Eq. (9) can be written in terms of differential operator form as

$$\sum_{n=1}^{2}\alpha_n D^n a\left(x,t\right) = \sum_{m=0}^{2}\beta_m D^m \sigma\left(x,t\right) \tag{10}$$

where, the order m and n of sums on right hand side and left hand side in the Eq. (10) depends upon the structure of the mechanical model representing the viscoelastic body. α_n and β_m are the combinations of the material constants and $\alpha_2 = G_1, \alpha_1 = G_1 R_2$, $\beta_2 = 1$, $\beta_1 = R_1 + R_2$, $\beta_0 =$ $R_3 + R_4$, $D \equiv \frac{d}{dt}$. Eq. (10) is the required differential operator form of constitutive relation for the model for viscoelastic material to be studied.

Behavior of the model can be discussed as

I. On comparing the 2nd order derivatives, we get

$$\sigma = G_1 a \tag{10a}$$

Eq. (10a) implies that the instantaneous behavior is elastic due to spring G_1 only in section-1of the model.

II. On comparing Fist order derivatives, we get

$$\left\{\left(\theta_{12'} + \theta_{13}\right) + \left(\theta_{22} + \theta_{22'}\right)\right\}\sigma = \left(\theta_{22} + \theta_{22'}\right)G_1 a\sigma = G_1 a \tag{10b}$$

From Eq. (10b), the behavior is elastic but effect of viscosity of dashpots $\eta_2, \eta_{2'}$ and η_3 is apparent due to the presence of relaxation time θ_{ij} .

III. On comparing zeroth order derivatives, we get

$$\left(\theta_{12'}.\theta_{22} + \theta_{13}\left(\theta_{22} + \theta_{22'}\right)\right)\sigma = 0 \tag{10c}$$

The zeroth order derivatives show that the model is viscoelastic in nature. Hence, from the above discussion, it is clear that at the start the higher order derivates of time dominate, but as the time passes their dominancy decreases.

Governing Equations For Five Parameter Viscoelastic Model

One of the governing equation for the viscoelastic model is constitutive relation and is (Lakes (1998))

$$f\left(\sigma, \sigma,_{t,}\ \sigma_{,tt},\ \ \gamma, \gamma,_{t}\right) = 0 \tag{11}$$

$$\beta_2 \sigma_{,tt} + \beta_1 \sigma_{,t} + \beta_0 \sigma = \alpha_2\, \gamma_{,tt} + \alpha_1 \gamma_{,t} \tag{12}$$

The equation of motion is

$$\sigma_{,x} = \rho u_{,tt} \tag{13}$$

The displacement-strain relation is

$$a = u_{,x} \tag{14}$$

From Eq. (13) and Eq. (14), we can write

$$\left(\frac{1}{\rho}\sigma_{,x}\right)_{,x} = \frac{1}{\rho}\left[\sigma_{,xx} - (log\rho)_{,x}\,\sigma_{,x}\right] = (u_{,x})_{,tt} = a_{,tt} \tag{15}$$

Using Eq. (12), Eq. (13), Eq. (14) and Eq. (15) the shearing stress field is

$$\beta_2\sigma_{,ttt} + \beta_1\sigma_{,tt} + \beta_0\sigma_{,t} = \frac{\alpha_2}{\rho}\left\{\sigma_{,xxt} - (log\rho)_{,x}\,\sigma_{,xt}\right\} + \frac{\alpha_1}{\rho}\left\{\sigma_{,xx} - (log\rho)_{,x}\,\sigma_{,x}\right\} \tag{16}$$

Solution For Five Parameter Viscoelastic Model

We assume that the solution σ *(x, t)* of Eq. (16) may be represented by the Friedlander Series (1947)

$$\sigma(x,t) = \sum_{n=0}^{\infty} A_n F_n\left(t - h(x)\right). = \sum_{n=0}^{\infty} A_n F_n(x) \quad , x = t - h(x)\,, A_n = 0\,, n<0 \tag{17}$$

With, $F'_n = F_{n-1}\,, F_{n,x} = -h_{,x} F_{n-1}\,,\quad F_{n,t} = F_{n-1}$.

The various derivatives stress with respect to x and t are

$$\sigma = A_n F_n\;, \sigma_{,t} = A_n F_{n-1}, \sigma_{,tt} = A_n F_{n-2}, \sigma_{,ttt} = A_n F_{n-3},$$

$$\sigma_{,x} = A'_n F_n - h_{,x} A_n F_{n-1}\,,$$

$$\sigma_{,xx} = A''_n F_n - \left(2h_{,x} A'_n + h_{,xx} A_n\right) F_{n-1} + A_n h_{,x}^2 F_{n-2}$$

$$\sigma_{,xxt} = A''_n F_{n-1} - \left(2h_{,x} A'_n + h_{,xx} A_n\right) F_{n-2} + A_n h_{,x}^2 F_{n-3}$$

$$\sigma_{,xt} = A'_n F_{n-1} - h_{,x} A_n F_{n-2}\,, \tag{18}$$

From Eq. (17) and Eq. (18)

$$\beta_0 A_n F_{n-1} + \beta_1 A_n F_{n-2} + \beta_2 A_n F_{n-3} = \frac{\alpha_1}{\rho}\left\{A''_n - (log\rho)_{,x}\; A'_n\right\} F_n + \begin{bmatrix} \dfrac{\alpha_1}{\rho}\begin{Bmatrix} h_{,x}(log\rho)_{,x} A_n \\ -\begin{pmatrix} 2h_{,x} A'_n \\ +h_{,xx} A_n \end{pmatrix} \end{Bmatrix} \\ +\dfrac{\alpha_2}{\rho}\begin{Bmatrix} A''_n \\ -(log\rho)_{,x}\; A'_n \end{Bmatrix} \end{bmatrix} F_{n-1} + \begin{bmatrix} \dfrac{\alpha_1}{\rho} h_{,x}^2 A_n \\ +\dfrac{\alpha_2}{\rho}\begin{Bmatrix} (log\rho)_{,x} h_{,x} A_n \\ -\begin{pmatrix} 2h_{,x} A'_n \\ +h_{,xx} A_n \end{pmatrix} \end{Bmatrix} \end{bmatrix} F_{n-2} + \frac{\alpha_2}{\rho} A_n h_{,x}^2 F_{n-3} \tag{19}$$

Comparing the coefficient of F_n , we get

$$\frac{\alpha_1}{\rho}\left\{A''_n - (log\rho), x\;\, A'_n\right\} = 0 \qquad A''_n = (log\rho)_{,x}\, A'_n \tag{20}$$

Comparing the coefficient of F_{n-1}, we get

$$\beta_0 A_n = \left[\frac{\alpha_1}{\rho} \left\{ h_{,x} (log\rho)_{,x} A_n - \left(2h_{,x} A'_n + h_{,xx} A_n \right) \right\} + \frac{\alpha_2}{\rho} \left\{ A''_n - \left(log\rho \right)_{,x} A'_n \right\} \right] \tag{21}$$

Comparing the coefficient of F_{n-2}, we get

$$\beta_1 A_n = \frac{\alpha_1}{\rho} h_{,x}^2 A_n + \frac{\alpha_2}{\rho} \left\{ (log\rho)_{,x} h_{,x} A_n - \left(2h_{,x} A'_n + h_{,xx} A_n \right) \right\} \tag{22}$$

Comparing the coefficient of F_{n-3}, we get

$$\beta_2 A_n = \frac{\alpha_2}{\rho} A_n h_{,x}^2 \tag{23}$$

By putting the value of β_2 and α_2 in Eq. (23), we get

$$h_{,x}^2 = \frac{\rho}{G_1} \tag{24}$$

From Eq. (20) and Eq. (21), we get

$$A_n \beta_0 = \frac{\alpha_1}{\rho} \left\{ h_{,x} (log\rho)_{,x} A_n - \left(2h_{,x} A'_n + h_{,xx} A_n \right) \right\} \tag{25}$$

From Eq. (21) and Eq. (25)

$$\beta_0 \alpha_2 = \beta_1 \alpha_1 - \frac{\alpha_1^2}{G_1} \tag{26}$$

By putting the values of $\beta_0, \beta_1, \alpha_1, \alpha_2$ in Eq. (27), we get

$$(R_3 + R_4) G_1 = G_1 . R_2 \ (R_1 + \ R_2 \ ) - \frac{G_1^2}{G_1} R_2^2 \tag{27}$$

Finally we get,

$$R_1 R_2 = R_3 + R_4 \tag{28}$$

Eq. (28) is the expression for relaxation time for five parameter viscoelastic model.

Dynamic Loading And Its Justification

The time parameter $'t'$ is introduced into an experimental scheme in dynamic experiments by cyclic deformation of the specimen, frequency $'\omega'$ of the oscillations plays the role of the time factor. The cyclic deformation is the fundamental process of determining the mechanical characteristics. The greatest preference is given to harmonic oscillations. A Harmonic action of the stress/strain produces a corresponding harmonic response in the strain/stress. Let us consider that shearing strain $'a'$, induced in elastic body which can be expressed by a harmonic action as:

$$a = a_0 \ e^{iwt} \tag{29}$$

Where, a_0 is the amplitude, ω is the frequency of oscillations and $'t'$ is the time.

According to Hooke's law, stress $'\sigma'$ is

$$\sigma = \sigma_0 e^{iwt} \tag{30}$$

Where, $\sigma_0 = G_0 a_0$, at t=0

For an elastic body, the strain and stress vary harmonically and there is no lack in the harmonic motion in phase as both have e^{iwt} as a factor. Thus an elastic body responds instantaneously to the external action (strain/stress). The phase shift angle between strain and stress is zero. For an ideal viscous body, the Newton's law of flow to a fluid body is as:

$$\sigma = \eta_0 a_0 i\omega e^{iwt} = \sigma_0 e^{i\left(\omega t + \frac{\pi}{2}\right)} \tag{31}$$

Where, η_0 is the viscosity of the body and $\eta_0 a_0 \omega = \sigma_0$ at t=0. Thus, for a viscous deformation, stress advances by the strain by a phase angle $\frac{\pi}{2}$. Thus the phase shift angle for the stress-strain under periodic harmonic deformation for elastic body is zero and for the viscous body, it is $\frac{\pi}{2}$ Therefore the phase shift angle 'δ' for the viscoelastic body must be between zero and $\frac{\pi}{2}$ i.e. $0 < \delta < \frac{\pi}{2}$. The lagging in phase of the strain behind the stress is due to the presence of relaxation processes in the case of viscoelastic body, as phase shift angle δ, is given by $0 < \delta < \frac{\pi}{2}$. Hence, $a = a_0 \; e^{iwt}$ and $\sigma = \sigma_0 e^{i(wt+\delta)}$

If we represent the projection of the stress vector on axis of co-ordinates by taking $\sigma' \leftrightarrow x$ and $\sigma'' \leftrightarrow y$, where σ' and σ'' represent that real and imaginary parts respectively. If the strain is initially set harmonically, then the modulus of viscoelastic body with harmonic loading can be written as

$$\frac{\sigma^*}{a^*} = \frac{\sigma^*}{a'} = \frac{\sigma'}{a'} + i\frac{\sigma''}{a'} = G' + iG'' = G^* \tag{32}$$

The phase angle δ is given as

$$\tan\delta = \frac{G''}{G'} \tag{33}$$

In the case of present model (Five-Parameter; two springs $S_1(G_1), S_2(G_2)$; three dash-pots $D_2(\eta_2), D_{2'}(\eta_{2'}), D_3(\eta_3)$ which represents a linear viscoelastic behavior under a given action of loading, the stress is directly proportional to strain. This is also true for time dependent stress and strain relation i.e. for viscoelastic body, the stress is

$$\sigma(t) = G^* a_0 e^{iwt} \tag{34}$$

Where, $a = a_0 \; e^{iwt}$

Using, the relation $\sigma = Ga$ for an elastic body, the constitutive relation for the physical state representing the five parameter model is given by Eq. (9).

Using Eq. (32), Eq. (34) in Eq. (9), we get

$$\left\{-\omega^2 + (R_1 + R_2) i\omega + (R_3 + R_4)\right\} G^* a_0 \; e^{iwt} = G_1 \left(-\omega^2 + R_2 \omega i\right) a_0 \; e^{iwt} \tag{35}$$

On solving, we get

$$G^* \; (i\omega) \; = \; \frac{G_1 \left(-\omega^2 + R_2 \omega i\right)}{\left(\left(R_3 + R_4 - \omega^2\right) + \left(R_1 + R_2\right) i\omega\right)} \tag{36}$$

$$G^* \; (i\omega) = \frac{G_1 \left(-\omega^2 + R_2 \omega i\right) \left[\left(\left(R_3 + R_4 - \omega^2\right) - i\left(R_1 + R_2\right) i\omega\right)\right]}{A_1} \tag{37}$$

$$G' + iG'' = G'' \left(i\omega\right) = \frac{G_1 \left[\left\{R_2 \left(R_1 + R_2\right) - \left(R_3 + R_4\right) + \omega^2\right\} \omega^2 + \left\{R_2 \left(R_3 + R_4\right) + R_1 \omega^2\right\} \omega i\right]}{A_1} \tag{38}$$

Separate Eq. (38) into real and imaginary parts, we get

$$G' = \frac{G_1 \left[\left\{R_2 \left(R_1 + R_2\right) - \left(R_3 + R_4\right) + \omega^2\right\} \omega^2\right]}{A_1} \tag{39}$$

$$G'' = \frac{G_1 \left[\left\{R_2 \left(R_3 + R_4\right) + R_1 \omega^2\right\} \omega\right]}{A_1} \tag{40}$$

And loss tangent is given by

$$\tan\delta = \frac{G''}{G'} = \frac{R_2 \left(R_3 + R_4\right) + R_1 \omega^2}{\omega \left\{R_2 \left(R_1 + R_2\right) - \left(R_3 + R_4\right) + \omega^2\right\}} = \frac{R_2 \left(R_3 + R_4\right) + R_1 \omega^2}{\omega \left[\omega^2 - \left\{\left(R_3 + R_4\right) - R_2 \left(R_1 + R_2\right)\right\}\right]} \tag{41}$$

To find the values of G'',

we put $A = R_2 \left(R_3 + R_4\right)$, $B = R_1$, $E = R_1 + R_2$, $A_1 = \left(C - \omega^2\right)^2 + E^2 \omega^2.$, in Eq. (40), hence

$$G'' = G_1 \; \frac{A\omega + B\omega^3}{\left(C - \omega^2\right)^2 + E^2 \omega^2} \tag{42}$$

Where,

Now by taking

$$D\left(G^{''}(i\omega)\right)=0 \tag{43}$$

Where $D=\dfrac{d}{dt}$

With the help of Eq. (42) and Eq. (43) we get

$$\left(A+3B\omega^2\right)A_1-2\omega^2\left(A+B\omega^2\right)\left\{\left(E^2-2C\right)+2\omega^2\right\}=0$$

$$\left(A+3B\omega^2\right)\left(\left(C-\omega^2\right)^2+E^2\omega^2\right)-2\omega^2\left(A+B\omega^2\right)\left\{\left(E^2-2C\right)+2\omega^2\right\}=0$$

$$\left(A+3B\omega^2\right)\left\{C^2+\omega^4+\left(E^2-2C\right)\omega^2\right\}-2\omega^2\left[A\left(E^2-2C\right)+\left\{2A+B\left(E^2-2C\right)\right\}\omega^2+2B\omega^4\right]=0$$

$$AC^2+\left\{3BC^2-A\left(E^2-2C\right)\right\}\omega^2+\left\{-3A+B\left(E^2-2C\right)\right\}\omega^4-B\omega^6=0$$

$$\omega^6-\left\{E^2-\left(3\frac{A}{B}+2C\right)\right\}\omega^4+\left\{\frac{A}{B}E^2-\left(2\frac{A}{B}+3C\right)C\right\}\omega^2-\frac{AC^2}{B}=0 \tag{44}$$

Eq. (44) gives the dispersion equation for wave propagation. It is a cubic in ω^2, giving three roots, then either it has one real root and other complex roots as complex roots always occur in conjugate pairs or all the three roots are real and at $G^{''}$ has either a maximal value or minimum value at these roots.. Therefore, taking roots as $\omega_1^2,\omega_2^2,\omega_3^2$, we get

Sum of roots,

$$\omega_1^2+\omega_2^2+\omega_3^2=E^2-\left(3\frac{A}{B}+2C\right) \tag{45}$$

Product of roots taken two at a time,

$$\omega_1^2\omega_2^2+\omega_2^2\omega_3^2+\omega_3^2\omega_1^2=\frac{A}{B}E^2-\left(2\frac{A}{B}+3C\right)C \tag{46}$$

Products of roots,

$$\omega_1^2\omega_2^2\omega_3^2=\frac{AC^2}{B} \tag{47}$$

To determine $\omega_1^2,\omega_2^2,\omega_3^2$ through Eq. (47) seems not to be so easy, taking one of the root for the extreme value of $G^{''}$ as

$$\omega_1^2=C. \tag{48}$$

To find the other two roots ω_2^2,ω_3^2 for the Eq. (44) from Eq. (45), Eq. (46) and Eq. (47), such that

$$\omega_2^2+\omega_3^2=E^2-3\left(\frac{A}{B}+C\right) \tag{49}$$

$$\omega_2^2.\omega_3^2 = \frac{AC}{B} \tag{50}$$

Then Eq. giving ω_2^2, ω_3^2 can be expressed as

$$x^2 - \left\{E^2 - 3\left(\frac{A}{B} + C\right)\right\} x + \frac{AC}{B} = 0 \tag{51}$$

We get the roots,

$$x = \frac{1}{2}\left\{E^2 - 3\left(\frac{A}{B} + C\right)\right\} \pm \sqrt{E^2 - 3\left(\frac{A}{B} + C\right)^2 - 4\frac{AC}{B}} \tag{52}$$

$$\omega_2^2, \omega_3^2 = \frac{1}{2}\left\{E^2 - 3\left(\frac{A}{B} + C\right)\right\}\left[1 \pm \left\{1 - \frac{\frac{2AC}{B}}{\left(E^2 - 3\left(\frac{A}{B} + C\right)^2\right)^2}\right\}\right] \tag{53}$$

When $4AC < B\left\{E^2 - 3\left(\frac{A}{B} + C\right)\right\}^2 = \frac{1}{B}\left\{E^2 B - 3\left(A + BC\right)\right\}^2$

Taking –ve sign, $4ABC < \left\{E^2 B - 3(A + BC)\right\}^2$

$$\omega_2^2 = \frac{AC}{B\left\{E^2 - 3\left(\frac{A}{B} + C\right)\right\}} = \frac{AC}{BE^2 - 3\left(A + BC\right)} \tag{54}$$

$$\omega_3^2 = \frac{AC}{B\omega_2^2} = \frac{BE^2 - 3\left(A + BC\right)}{B} \tag{55}$$

Error due to approximation is

$$4AC << B\left\{E^2 - 3\left(\frac{A}{B} + C\right)\right\} \tag{56}$$

From, Eq. (46)

$$\omega_1^2\omega_2^2 + \omega_2^2\omega_3^2 + \omega_3^2\,\omega_1^2 = \frac{AE^2 - \left(2A + 3BC\right)C}{B} \tag{57}$$

Approximate value

$$\omega_1^2\omega_2^2 + \omega_2^2\omega_3^2 + \omega_3^2\,\omega_1^2 = \frac{AC^2}{\left\{E^2 B - 3\left(A + BC\right)\right\}} + \frac{AC}{B} + \frac{C}{B}\left\{E^2 B - 3\left(A + BC\right)\right\} \tag{58}$$

Error can be calculated by subtracting Eq. (57) and Eq. (58)

$$\text{Error }(\xi) = \frac{A}{B}E^2 - \frac{3(A+BC)C}{B} - \frac{AC^2}{\{E^2B-3(A+BC)\}} - \frac{C}{B}\{E^2B-3(A+BC)\} \tag{59}$$

$$(\xi) = \frac{1}{B}\{AE^2-3(A+BC)C\} - \frac{C}{B}\frac{\left[ABC+\{E^2B-3(A+BC)\}^2\right]}{E^2B-3(A+BC)} \tag{60}$$

Taking the +ve sign, we get

$$\omega_3^2 = \frac{1}{B}\{E^2B-3(A+BC)\}\left[1-\frac{ACB}{\{E^2B-3(A+BC)\}^2}\right]$$

$$\omega_3^2 = \frac{1}{B}\{E^2B-3(A+BC)\} - \frac{AC}{E^2B-3(A+BC)} \tag{61}$$

Case-1

At very low frequencies, $\omega = 0$, (from Eq. (42))

$$G''(\omega=0)=0, G''\left(\omega^2=-\frac{R_2(R_3+R_4)}{R_1}\right)=0 \tag{62}$$

Then it is to be inferred that during the cyclic loading initially $\omega = 0$ *i.e.* $G''(0)=0$, there must be a point of maxima or minima between $\omega = 0$ and $\omega = -\frac{R_2(R_3+R_4)}{R_1}$.

Case-2

Also,

$$G''(\omega^2=R_3+R_4)=G''_m=\frac{G_1\sqrt{(R_3+R_4)}}{(R_1+R_2)} \tag{63}$$

At very high frequencies, $\omega = \infty$ (from Eq. (42))

$$G''(\infty)=0 \tag{64}$$

But for $\omega^2 = R_3+R_4$ it is observed that for $\omega^2 = C$ there must be a point of maxima as when initially G'' (0) increases from zero to maximum value $G''_m = \frac{G_1\sqrt{(R_3+R_4)}}{(R_1+R_2)}$ and again states that diminishing and reaches zero at $\omega^2 = -\frac{R_2(R_3+R_4)}{R_1}$, which justifies the model for the Viscoelastic materials. When relaxation is applied to the model i.e. the model is under the influence of constant deformation, the specimen representing the model is deformed to the given strain a_0 and after which it is maintained constant, where as the stress required to maintained these strains (a_0) diminishes at $a = a_0$ constant, under thermal conditions. The Constitutive equation under constant deformation (strain) $a = a_0$ constant reduces to

$$\beta_2 \sigma_{,tt} + \beta_1 \sigma_{,t} + \beta_0 \sigma = 0 \tag{65}$$

Where,

$$\beta_2 = 1, \ \beta_1 = \left\{ \left(\theta_{12'} + \theta_{13} \right) + \left(\theta_{22} + \theta_{22'} \right) \right\}, \ \beta_0 = \ \theta_{12'} . \theta_{22} + \theta_{13} \left(\theta_{22} + \theta_{22'} \right)$$

Eq. (65) can be solved, we taking the roots of the auxiliary equation as

$$m_1 = \frac{1}{\tau_1} \text{ and } m_2 = \frac{1}{\tau_2} . \tag{66}$$

Where, τ_1 and τ_2 are relaxation times of the specimen.

$$m_1 + m_2 = \frac{1}{\tau_1} + \frac{1}{\tau_2} = \left(\theta_{12'} + \theta_{13} \right) + \left(\theta_{22} + \theta_{22'} \right) \tag{67}$$

$$m_1 m_2 = \frac{1}{\tau_1} . \frac{1}{\tau_2} = \theta_{12'} \theta_{22} + \theta_{13} (\theta_{22} + \theta_{22'}) \tag{68}$$

From, Eq. (67) and Eq. (68), the Eq. (65) becomes

$$\left\{ D^2 + \left(m_1 + m_2 \right) D + (m_1 m_2) \right\} \sigma = 0 \tag{69}$$

The Solution of Eq. (69) is

$$\sigma(t) = A_1 e^{-m_1 t} + A_2 e^{-m_2 t} \tag{70}$$

To eliminate A_1 and A_2

At, $t = 0$, Eq. (70) reduces to $\sigma_0 = a_0 G, \ \dfrac{da_0}{dt} = 0$

Hence,

$$A_1 + A_2 = \sigma_0 \tag{71}$$

$$m_1 A_1 + m_2 A_2 = 0 \tag{72}$$

Where,

$$A_1 = \frac{m_2 \sigma_0}{m_2 - m_1} \quad , A_2 = \frac{m_1 \sigma_0}{m_1 - m_2}$$

$$\therefore \sigma(t) = \frac{\sigma_0}{m_1 - m_2} \left\{ m_1 e^{-m_2 t} - m_2 e^{-m_1 t} \right\} = \frac{G_0 \gamma_0}{m_1 - m_2} \left\{ m_1 e^{-m_2 t} - m_2 e^{-m_1 t} \right\} \tag{73}$$

For sufficiently large time, $t > \tau_{1,}\tau_2$ so that $\sigma \to 0$. Hence with longer periods of observation, the stress in the specimen will drop to zero, i.e. equilibrium state will be achieved.

Complex Viscosity

Let, here not the strain is specified but the stress which varies by the harmonic law:

$$\sigma = \sigma_0 e^{i\omega t} \tag{74}$$

Let the model instantly respond to the change in stress by undergoing a strain equal to $I_0\sigma$, which occur in phase with $\sigma(t)$ .Where I_0 is the instantaneous compliance

A viscous flow develops by the law

$$\frac{da}{dt}\eta = \sigma(t) \tag{75}$$

On integrating, we get

$$a = \frac{\sigma_0}{i\omega\eta}e^{i\omega t} = -i\frac{1}{\omega\eta}\sigma(t) \tag{76}$$

Therefore, there exist third strain components, which is out of phase with the specified change in stress. Now, the strain $a(t)$ is given by

$$a(t) = I_0\sigma(t) + \gamma_0^{'} e^{i(\omega t - \delta)} - i\frac{1}{\omega\eta}\sigma(t) \tag{77}$$

where, $\gamma_0^{'}$ is the amplitude of strain, which is out of phase with $\sigma(t)$ by the phase angle δ .

Let I^* be the complex compliance as the inverse of complex modulus G^* such that $G^* \; I^* = 1$, hence

$$I^* = \frac{a(t)}{\sigma(t)} \; or, I^* = \left(I_0 - i\frac{1}{\omega\eta}\right) + \frac{a_0^{'}}{\sigma_0}e^{-i\delta} \quad \Rightarrow I^* = \left(I_0 + \frac{a_0^{'}}{\sigma_0}\cos\delta\right) - i\left(\frac{1}{\omega\eta} + \frac{a_0^{'}}{\sigma_0}\sin\delta\right)$$

$$\therefore I^* = \left(I_0 + I^{'}\right) - i\left(\frac{1}{\omega\eta} + I^{''}\right) \tag{78}$$

Where $I^{'} = \frac{a_0^{'}}{\sigma_0}\cos\delta; I^{''} = \frac{a_0^{'}}{\sigma_0}\sin\delta$

So it is clear that for I^* the quantity I_0 corresponds to the instantaneous elastic deformations of the material, and $\frac{1}{\omega\eta}$ corresponds to viscous flow. Therefore, the viscoelastic behavior of the material is governed by the values of $I^{'}$ and $I^{''}$. Assuming that $I_0 \ll I^{'}$ and $\frac{1}{\omega\eta} \ll I^{''}$ .Using relation $G^* I^* = 1$, the relation between complex modulus and complex compliance is obtained as:

$$\begin{aligned} G^{'} &= \frac{I^{'}}{\left(I^{'}\right)^2 + \left(I^{''}\right)^2}; G^{''} = \frac{I^{''}}{\left(I^{'}\right)^2 + \left(I^{''}\right)^2} \\ I^{'} &= \frac{G^{'}}{\left(G^{'}\right)^2 + \left(G^{''}\right)^2}; I^{''} = \frac{G^{''}}{\left(G^{'}\right)^2 + \left(G^{''}\right)^2} \end{aligned} \tag{79}$$

Using Eq. (39) and Eq. (40), we get the complex compliance for five parameter viscoelastic model. The tangent of the angle δ (the loss factor) is expressed in terms of the ratio of the components of the complex compliance as well as in terms of the ratio $\tan\delta = \frac{I^{''}}{I^{'}}$. When the sinusoidally varying stress is specified, the change in the rate of strain can be followed such that

$$\dot{a} = \frac{da}{dt} = a_0 i\omega e^{i(\omega t - \delta)} = i\omega a \tag{80}$$

Then we define the complex viscosity

$$\eta^* = \frac{\sigma}{\dot{a}} = \frac{\sigma_0 e^{i\delta}}{a_0 i\omega} = \frac{\sigma_0}{a_0\omega}\left(\sin\delta - i\cos\delta\right) = \eta^{'} - i\eta^{''} \tag{81}$$

Where

$$\eta^{'} = \frac{\sigma_0}{a_0\omega}\sin\delta\,, \eta^{''} = \frac{\sigma_0}{a_0\omega}\cos\delta$$

Here $a_0\omega$ is the amplitude of the strain rate.

Since $$G^* = \frac{\sigma(t)}{a(t)} = G^{'} + G^{''}$$

where $G^{'} = \frac{\sigma_0^{'}}{a_0}\cos\delta; G^{''} = \frac{\sigma_0^{'}}{a_0}\sin\delta$

Using the above relations, we get the relation between complex viscosity and complex modulus as:

$$\eta^{'} = \frac{G^{''}}{\omega}; \eta^{''} = \frac{G^{'}}{\omega} \tag{82}$$

The quantity $\eta^{'}$ is often for simplicity called just the dynamic viscosity.

Constitutive Relation For Four Parameter Model

We consider the four parameter model with two springs $S_1(G_1), S_2(G_2)$ and two dash-pots $D_1(\eta_1), D_2(\eta_2)$ with viscoelasticity η_1 and η_2 respectively (Fig.1). The springs represent recoverable elastic response and dash pot represents elements in the structure giving rise to viscous drag. Let a_1 be the strain in $S_1(G_1)$, a_2 be the strain along dashpot $D_1(\eta_1)$ and a_3 be the strain in the Kelvin model. Fig. 1 represents the sketch of the standard four parameter viscoelastic models. The stress v/s strain behavior for constant stress (σ) with time (t_a) has been shown in fig. 1. Here, $G_1 = \lambda_1 + 2\mu_1$, $G_2 = \lambda_2 + 2\mu_2$ are the modulli of elasticity, η_1, η_2 are Newtonian viscosities coefficients and taken as functions of 'x' in the non-homogeneous case.

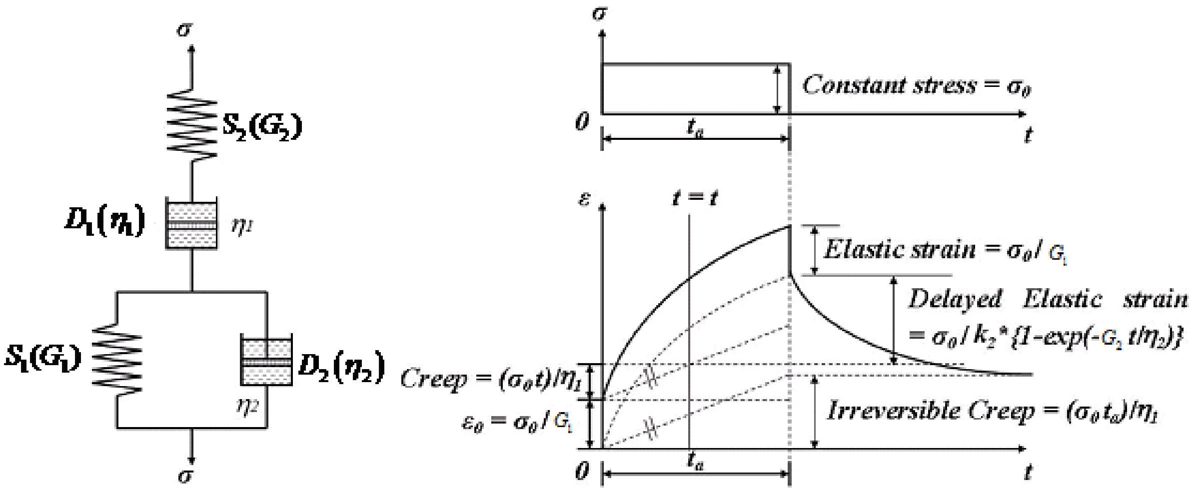

Figure-2 Rheological model and its response

The stress strain equation for four parameter model is of general form. A four parameter model consisting of two springs and two das-pots may be regarded as a Maxwell unit in series with a Kelvin unit as illustrated in Fig. 1. Let σ is stress and 'a' is shear strain; the relations between them are given as

$$a = a_1 + a_2 + a_3, \sigma = G_1 a_1, \sigma = \eta_1 \dot{a}_2, \sigma = G_2 a_3, \sigma = \eta_2 \dot{a}_3 \tag{83}$$

Eliminating $a_1, a_{2,} a_3$, we get the constitutive Eq. (1) as

$$\ddot{\sigma} + \left(\frac{G_1}{\eta_2} + \frac{G_2}{\eta_2} + \frac{G_1}{\eta_1} \right) \dot{\sigma} + \frac{G_1 G_2}{\eta_1 \eta_2} \sigma = G_1 \ddot{a} + \frac{G_1 G_2}{\eta_2} \dot{a} \tag{84}$$

Dynamic Response Of Four Parameter Viscoelastic Model

The following values of strain are taken for the dynamical response of four parameter model

$$\sigma = G^* a_0 e^{iwt}, a = a_0 e^{iwt} \tag{85}$$

From Eq. (84) and Eq. (85), we get

$$\left\{-\omega^2 + \left(\frac{G_1}{\eta_2} + \frac{G_2}{\eta_2} + \frac{G_1}{\eta_1}\right) i\omega + \frac{G_1 G_2}{\eta_1 \eta_2}\right\} G^* = G_1 \left\{-\omega^2 + \left(\frac{G_1 G_2}{\eta_2}\right) i\omega\right\} \tag{86}$$

On simplifying Eq. (86), we get

$$G^* = \frac{G_1\left(-\omega^2 + R_8 i\omega\right)\left\{\left(R_9 - \omega^2\right) - \left(R_5 + R_6 + R_7\right) i\omega\right\}}{A_2} \tag{87}$$

where, $A_2 = \left(R_9 - \omega^2\right)^2 + \left(R_5 + R_6 + R_7\right)^2 \omega$

G^* can be written in terms of real and imaginary parts

$$G^* = \frac{G_1\left[\left(\omega^2 - R_9\right)\omega^2 + \left\{R_8\left(R_5 + R_6 + R_7\right)\right\}\omega^2\right] + i\left\{R_8\left(R_9 - \omega^2\right)\omega + R_8\left(R_5 + R_6 + R_7\right)\omega^3\right\}}{A_2} \tag{88}$$

$$\text{or } G^* = G' + G'' \tag{89}$$

$$G' = \frac{G_1\left[\left(\omega^2 - R_9\right)\omega^2 + \left\{R_8\left(R_5 + R_6 + R_7\right)\right\}\omega^2\right]}{A_2}, \tag{90a}$$

$$G'' = \frac{\left\{R_8\left(R_9 - \omega^2\right)\omega + R_8\left(R_5 + R_6 + R_7\right)\omega^3\right\}}{\left(R_9 - \omega^2\right)^2 + \left(R_5 + R_6 + R_7\right)^2 \omega}. \tag{90b}$$

The loss tangent is,

$$\tan\delta = \frac{G''}{G'} \tag{91}$$

From Eq. (90a), Eq. (90b) and Eq. (91), we get

$$\tan\delta = \frac{R_8\left(R_9 - \omega^2\right) + R_8\left(R_5 + R_6 + R_7\right)\omega}{\left(\omega^2 - R_9\right) + \left\{R_8\left(R_5 + R_6 + R_7\right)\right\}} \tag{92}$$

Numerical Analysis

The behavior of both the models have been studied numerically as well as graphically, the rheological responses are discussed by plotting a graph between G'' and ω and G' verses ω for five parameter and four parameter models.

For five parameter model, we have

$$G'' = G_1 \frac{A\omega + B\omega^3}{\left(C-\omega^2\right)^2 + E^2\omega^2}.$$

Here,

$A = R_2\left(R_3+R_4\right),\ B = R_1,\ C = R_3+R_4,\ E = R_1+R_2,\ R_1 = \theta_{12'}+\theta_{13},\ R_2 = \theta_{22}+\theta_{22'},\ R_3 = \theta_{12'}.\theta_{22}$ and $R_4 = \theta_{13}R_2$.

To calculate G'' at different frequencies, we assume the following values

$$G_1 = 1.0,\ \ G_2 = 1.10,\ \eta_2 = 0.1,\ \eta_{2'} = 0.2,\ \eta_3 = 0.3$$

Using these values we get

$$R_1 = 8.33, R_2 = 16.5, R_3 = 55, R_4 = 54.945, A = 1814.175, B = 8.33, C = 109.945, E = 24.83$$

Now,

$$G' = \frac{G_1\left[\left\{R_2\left(R_1+R_2\right) - \left(R_3+R_4\right) + \omega^2\right\}\omega^2\right]}{A_1}$$

where,

$$R_1 = 8.33, R_2 = 16.5, R_3 = 55, R_4 = 54.945,\ A_1 = \left(C-\omega^2\right)^2 + E^2\omega^2$$

Fig. (3-4), has been plotted for five parameter model. It is quite clear from graph, form fig. 3 and fig. 4 for ω =7, there is peak for the graphs G'' verse ω and G' verses ω, as the value of ω increases both G'' and G' decreases i.e the exponential decay takes place. However, the decay in fig. 4 is steeper as compared in fig. 3.

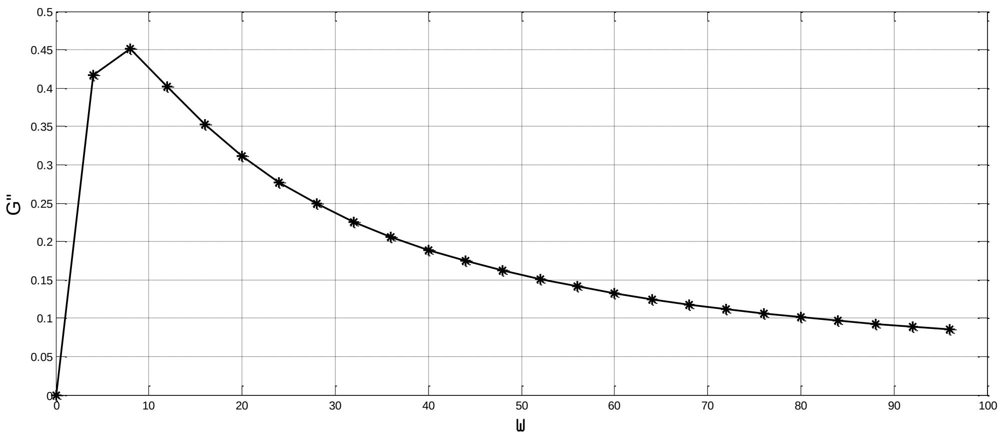

Figure-3 Variation of G'' verses ω for five parameter model

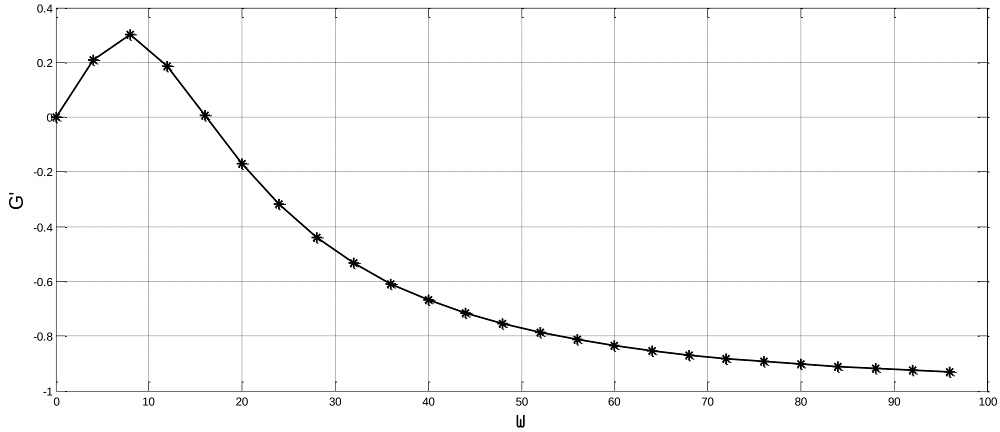

Figure-4 Variation of $G^{'}$ verses ω for five parameter model

Four parameter model, we have

$$G^{''}=\frac{\left\{R_8\left(R_9-\omega^2\right)\omega+\left\{R_8\left(R_5+R_6+R_7\right)\right\}\omega^3\right\}}{\left(R_9-\omega^2\right)^2+\left(R_5+R_6+R_7\right)^2\omega}.$$

The values of the parameters for studying the rheological response are

$G_1=1.0,\quad G_2=1.10,\quad \eta_1=0.1,\quad \eta_2=0.2,\quad R_5=5,\quad R_6=5.5,\quad R_7=10,\quad R_8=5.5,\quad R_9=55.$

Now,

$$G^{'}=\frac{G_1\left[\left(\omega^2-R_9\right)\omega^2+\left\{R_8\left(R_5+R_6+R_7\right)\right\}\omega^2\right]}{\left(R_9-\omega^2\right)^2+\left(R_5+R_6+R_7\right)^2\omega}$$

A graph between $G^{'}$ verses ω is plotted by taking above equation. The parameters taken for this case are the same. The fig. (5-6), shows that as the value of ω increases both $G^{''}$ and $G^{'}$ decreases, but the value of $G^{'}$ becomes constant after ω=55 for four parameter model.

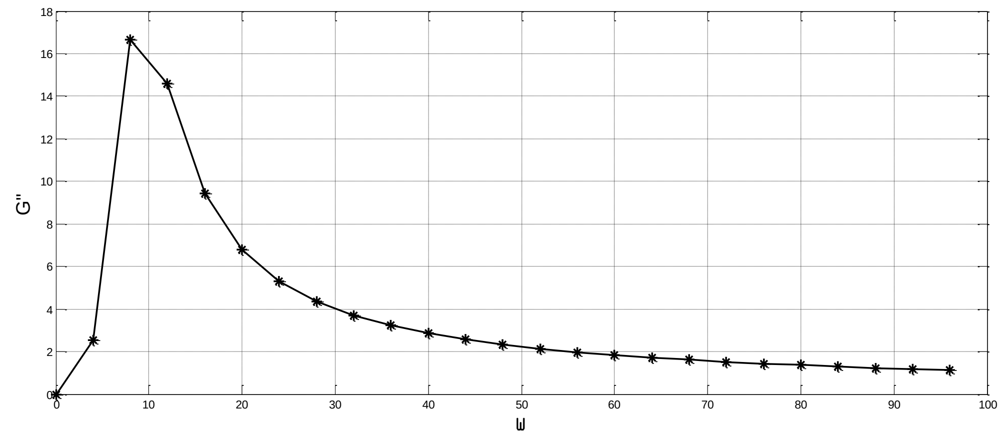

Figure-5 Variation of $G^{''}$ verses ω for four parameter model

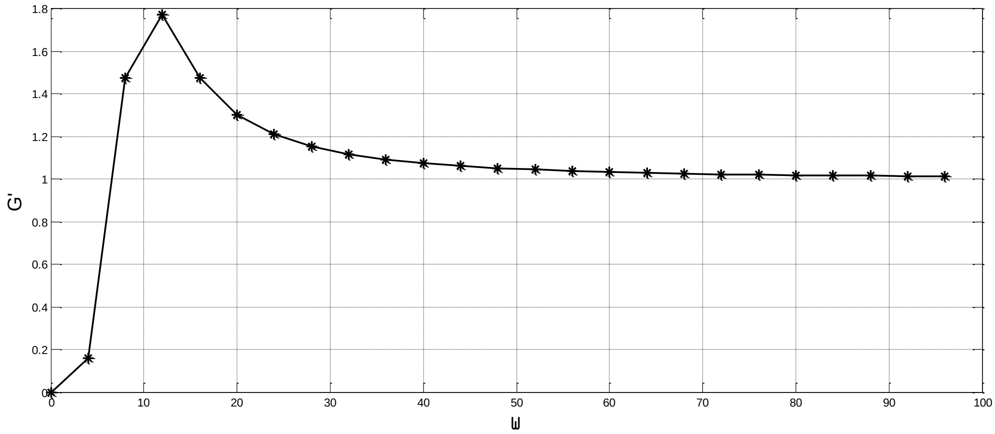

Figure-6 Variation of $G^{'}$ verses ω for four parameter model

Conclusions

It can be concluded that the both models possess an excellent potential for proper representation of the time dependent behavior of a viscoelastic medium subjected to loading and unloading. However, five parameter models are slightly better than four parameter model. The viscous strains due to a constant stress are found to increase linearly with time for both the models. Moreover, after the removal of stress, the viscous strain is found to remain constant with time. During the cyclic loading initially there must be a point of maxima or minima for five parameter model between $\omega = 0$ and $\omega = -\frac{\alpha_2(\alpha_3+\alpha_4)}{\alpha_1}$. For sufficiently large relaxation time for five parameter model, the stress in the specimen will drop to zero. The phase shift angle $'\delta'$ for the viscoelastic body for five parameter model must be between zero and $\frac{\pi}{2}$. The use of five parameter and four parameter models are mostly restricted in the field of rock mechanics. Thus, both models can be used in determining the time-dependent behavior of a viscoelastic medium.

Acknowledgements

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