MATHEMATICAL TREATMENT OF OSCILLATORY SYSTEMS USING THE FRACTIONAL CALCULUS

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Abstract

In this work, the fractional calculus methods are used to solve essential problems in conservative and non-conservative oscillatory systems. Regarding the non-conservative systems, the key factor is to modify the standard fractional Lagrange equations by including the fractional Rayleigh's dissipation function with a time fractional derivative of the displacement. The results are tested by applying them to well known Oscillatory systems under conservative and non-conservative forces. The calculations reveal that, the equations of motion are controlled by the fractional order derivative (alpha), as (alpha) goes to unity the equations of motion become as those for ordinary oscillatory systems.

Keywords: Fractional Calculus, Rayleigh's Dissipation Function, Oscillatory Systems, Lagrange-Equations of Motion

Introduction

The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695. Fractional calculus is employed in several fields: Mathematics, physics, engineering, biology, and other scientific fields [1-5].

Fractional calculus is a generalization of integration and differentiation to non-integer order, being the fundamental operator is ${}_aD_t^{\alpha}$ where α and *t* are the limits of the operation [6-9]. Fractional calculus was employed to describe several physical phenomena such as heat flow, electricity, magnetism, and fluid dynamics. As an example of that, the electromagnetic theory adopted the fractional calculus to describe the charge distribution of a dipole.

In last decade, many studies have brought fractional calculus into attention revealing that many physical phenomena are modeled by fractional differential equations [4-5]. The importance of fractional order mathematical models is that it can be used to make a more accurate prediction and to give a deeper insight into physical processes.

Riewe [10,11] constructed a complete mechanical description of nonconservative systems including Lagrangian and Hamiltonian mechanics, canonical transformations, Hamilton-Jacobi theory, and quantum wave mechanics by using fractional derivatives. He showed that the formalism can be applied to a classical fractional force proportional to the velocity.

On the other hand, Rabei *et al* [12] found a method to obtain potentials for nonconservative forcecs in order to introduce dissipative effects to the Lagrangian and Hamiltonian mechanics.

Recently, the fractional constrained Lagrangian and Hamiltonian were analyzed [13-14]. The notion of the fractional Hessian was introduced and the Euler-Lagrange equations were obtained for a Lagrangian linear in velocities.

Fractional-order circuits and systems have witnessed an increasing interest lately [15]. Capacitors are one of the crucial elements in integrated circuits and are used extensively in many of them, such as sample and holds, radio-frequency oscillators, mixers [16,17].

The paper is organized as follows: In section 2, fractional calculus of conservative forces is reviewed briefly. In section 3, the fractional calculus of nonconservative forces is introduced. Applications on conservative and nonconservative systems are introduced in sections 4 and 5 respectively. The paper closes with some concluding remarks in section 6.

Fractional Calculus of Conservative Forces

In Agrawal's work [18-20], the problem is formulated in terms of the left and right Riemann-Louville fractional derivatives, which are defined as:

The left Riemann-Louville fractional derivative reads as

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha-1} f(\tau)d\tau \tag{1} $$

which is denoted as the LRLFD and the right Riemann-Louville fractional derivative reads as

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n \int_x^b (\tau-x)^{n-\alpha-1} f(\tau)d\tau \tag{2} $$

which is denoted as the RRLFD. Here α is the order of the derivative such that $n-1 \le \alpha < n$ and is not equal to zero. If α is an integer, these derivatives are defined in the usual sense, i.e.,

$$ {}_aD_x^\alpha f(x) = \left(\frac{d}{dx}\right)^\alpha f(x); \quad {}_xD_b^\alpha f(x) = \left(-\frac{d}{dx}\right)^\alpha f(x); \qquad \alpha = 1,2,\ldots. \tag{3}$$

The Euler-Lagrange equations for the fractional calculus of variations problem is obtained as

$$\frac{\partial L}{\partial q} + {}_tD_b^\alpha \frac{\partial L}{\partial \, {}_aD_t^\alpha q} + {}_aD_t^\beta \frac{\partial L}{\partial \, {}_tD_b^\beta q} = 0 . \tag{4}$$

Here L is a function of the form

$$L = L(q, {}_aD_t^\alpha q, {}_tD_b^\beta q, t). \tag{5}$$

For $\alpha = \beta = 1$, we have ${}_aD_t^\alpha = \frac{d}{dt}$ and ${}_tD_b^\alpha = -\frac{d}{dt}$, and Eq.(4) reduces to the standard Euler-Lagrange equation.

Fractional Calculus of Nonconservative Forces

Another point regarding Lagrange's equations must be noted. Only if some of the forces acting on the system are derivable from the potential, can Lagrange's equations assume the form

$$\frac{\partial L}{\partial q} + {}_tD_b^\alpha \frac{\partial L}{\partial \, {}_aD_t^\alpha q} + {}_aD_t^\beta \frac{\partial L}{\partial \, {}_tD_b^\beta q} + Q_j = 0, \tag{6}$$

where the Lagrangian L contains only those forces that are conservative while Q_j includes the forces that are not derivable from potential. An illustration of this letter type of force is the frictional force that is proportional to fractional time derivative of position which may be written as

$$F_i = -k \, {}_aD_t^\alpha x_i , \tag{7}$$

where k is a constant.

Forces of this type are derivable from fractional Rayleigh's dissipative function, f, defined by

$$\mathrm{f} = \frac{1}{2} k \left({}_aD_t^\alpha x_i \right)^2 . \tag{8}$$

It is obvious that the frictional force can be written as

$$F_i = -\frac{\partial f}{\partial\ {}_aD_t^{\alpha} x_i}. \tag{9}$$

Component of Q_j of the generalised force arising as a result of fractional force is given by

$$Q_j = \sum_i F_i \frac{\partial\ {}_aD_t^{\alpha} x_i}{\partial\ {}_aD_t^{\alpha} q_j} = -\sum_i \frac{\partial f}{\partial\ {}_aD_t^{\alpha} x_i}.\frac{\partial\ {}_aD_t^{\alpha} x_i}{\partial\ {}_aD_t^{\alpha} q_j} = -\frac{\partial f}{\partial\ {}_aD_t^{\alpha} q_j}. \tag{10}$$

Substituting this value of Q_j into Eq.(6), we can write Lagrange's equation of motion as

$$\frac{\partial L}{\partial q} + {}_tD_b^{\alpha} \frac{\partial L}{\partial\ {}_aD_t^{\alpha} q} + {}_aD_t^{\beta} \frac{\partial L}{\partial\ {}_tD_b^{\beta} q} - \frac{\partial f}{\partial\ {}_aD_t^{\alpha} q} = 0. \tag{11}$$

Thus, if fractional forces of friction are acting on the system, we must specify two scalar functions – the fractional Lagrangian and fractional Rayleigh's dissipative function to derive the fractional equations of motion.

Applications on Conservative Systems

Harmonic Oscillator

As a first example of conservative systems consider a harmonic oscillator of stiffness k attached to a block of mass m. If the block is displaced a distance x from equilibrium, the fractional Lagrangian of this oscillatory system is

$$L = \frac{1}{2} m \left({}_aD_t^{\alpha} x\right)^2 - \frac{1}{2} k x^2. \tag{12}$$

Making use of Eq.(4), the fractional Lagrange's equation of motion can be obtained as

$$-kx + m\ {}_tD_b^{\alpha}\left({}_aD_t^{\alpha} x\right) = 0. \tag{13}$$

In the limit $\alpha \to 1$, equation (13) reduces to the equation of motion of the undamped harmonic oscillator:

$$m\ddot{x} + kx = 0.$$

U-tube

Consider now a shaped U-tube of length l filled with a liquid of density ρ, if the liquid in one level is initially displaced vertically a distance 2x from the other level, the fractional Lagrangian has the form

$$L = \frac{1}{2} \rho A l \left({}_aD_t^{\alpha} x\right)^2 - \rho A g x^2. \tag{14}$$

Following Eq.(4), the Lagrange's equation of motion reads

$$-2\rho Agx + \rho Al \ {}_tD_b^{\alpha}\left({}_aD_t^{\alpha} x\right) = 0 . \qquad (15)$$

Or

$$-2gx + l\,{}_tD_b^{\alpha}\left({}_aD_t^{\alpha} x\right) = 0 . \qquad (16)$$

If α goes to one, then we must require that

$$l\ddot{x} + 2gx = 0 .$$

Applications on Non-conservative Systems

Damped Harmonic Oscillator

As a first example of nonconservative systems consider the damped harmonic oscillator. The fractional Lagrangian and the fractional Rayleigh's dissipation function describing this motion are

$$L = \frac{1}{2} m\left({}_aD_t^{\alpha} x\right)^2 - \frac{1}{2} kx^2 ; \qquad f = \frac{1}{2} c\left({}_aD_t^{\alpha} x\right)^2 \qquad (17)$$

Substituting Eq.(17) into Eq.(11), we get

$$-kx + m \ {}_tD_b^{\alpha}\left({}_aD_t^{\alpha} x\right) - c\,{}_aD_t^{\alpha} x = 0 . \qquad (18)$$

In the limit $\alpha \to 1$, we obtain the equation of motion of the damped harmonic oscillator

$$m\ddot{x} + c\dot{x} + kx = 0 .$$

RL Circuit

Consider now an electric circuit consisting of a resistor, inductor, and battery. The fractional Lagrangian for this circuit is

$$L = \frac{1}{2} l\left({}_aD_t^{\beta} q\right)^2 - \varepsilon q , \qquad (19)$$

and the Rayleigh's fractional dissipation function is

$$f = \frac{1}{2} R\left({}_aD_t^{\alpha} q\right)^2 . \qquad (20)$$

The Lagrange's equation of motion reads

$$-\varepsilon + l \ {}_tD_b^{\alpha}\left({}_aD_t^{\alpha} q\right) - R \ {}_aD_t^{\alpha} q = 0 . \qquad (21)$$

For $\alpha \to 1$, we get the equation of the electrical driven forced oscillator in the form

$$l\ddot{q} + R\dot{q} + \varepsilon = 0 .$$

RLC Circuit

We now turn our attention to more realistic circuit consisting of an indicator, capacitor, and a resistor connecting in series. The fractional Lagrangian and the fractional Rayleigh's dissipation function for this circuit are

$$L = \frac{1}{2} l \left({}_a D_t^\alpha q \right)^2 - \frac{q^2}{2c}; \qquad f = \frac{1}{2} R \left({}_a D_t^\alpha q \right)^2. \tag{22}$$

The Lagrange's equation of motion can be obtained as

$$-\frac{q}{c} + l \; {}_t D_b^\alpha \left({}_a D_t^\alpha q \right) - R \; {}_a D_t^\alpha q = 0. \tag{23}$$

As $\alpha \to 1$, we arrive at the equation of the electrical damped harmonic oscillator

$$l\ddot{q} + R\,\dot{q} + \frac{q}{c} = 0.$$

Conclusion

As a result of this work, fractional calculus is a powerful method to solve mechanical energy issues related to oscillatory systems. For conservative oscillatory systems with ordinary potential energy and fractional kinetic energy, the Lagrangian's equations are obtained. Similarly, modified Lagrangian's equations are obtained for non-conservative oscillatory systems with quadratic time dependant fractional Rayleigh's dissipation function. For both conservative and non-conservative cases, the equations of motion result by this method return to the ordinary differential equations as the fractional order derivative (alpha) goes to unity.

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