UNIFORM STRUCTURES WITH CONTRACTIVE MAPPINGS AND PRODUCT OF FUZZY METRIC SPACES

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Abstract

According to the notion of a fuzzy metric due to George and Veeramnai, we study and extends some topological Properties to fuzzy metric spaces such as uniform continuity-uniformly convergence – equicontinuous sequences of functions, also some topological properties for product of fuzzy metric spaces with contractively and fixed point theorem are studied , some important and interesting results of their properties are obtained.

Keywords: Fuzzy metric spaces, uniformly continuous, equicontinuty, product fuzzy metric

Introduction and Preliminaries

In 1965, the concept of fuzzy set was introduced by Zadeh [21]. One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space ,this problem has been investigated by Many authors [4,5,7,8,13,14],they introduced the concept of fuzzy metric space in different ways. In particular George and Veeramani [8] have introduced and studied a notion of fuzzy metric space with the help of continuous t-norms, which constitutes a slight but appealing modification of the one due to Kramosil and Michalek [14] and defined a Hausdorff topology on this fuzzy metric space. In [1]we study the zero-dimensionality and small inductive dimension in fuzzy metric spaces . Many authors have studied fixed theory in fuzzy metric spaces such as [2,3,10,11,12,13,19] . Also R. Mohd, and S. Mohd, in [16] have introduced and studied a concept of Product of fuzzy metric.

The aim of this paper is to extend some concepts to fuzzy metric spaces such as (uniformly continuous and isometry of mappings between fuzzy metric spaces-convergence uniformly and equicontinuous of sequences of functions from fuzzy metric space to other, we obtain some results about them),Also we investigate some topological properties for Product of fuzzy metric spaces with fixed point property, some results about all concepts are given.

We now recall some notation and basic definitions used in this paper

Definition1.1[18] A binary operation $\ast:[0,1]\times[0,1] \to [0,1]$ is a continuous triangular norm (shortly t-norm) if $\ast$ satisfies the following conditions:

1. $\ast$ is associative and commutative.
2. $\ast$ is continuous.
3. $a\ast 1= a$ for all $a\in [0,1]$.
4. $a\ast b \le c\ast d$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d\in [0,1]$.

Example1.2 The following are examples of t-norm:

(1). $a \ast b = ab$. (2) $a \ast b = \min \{a, b\}$. (3) $a*b = \max \{0,a+b-1\}$, for all

$a, b \in [0, 1]$

Definition1.3[8] A fuzzy metric space is an ordered triple $(X,M,\ast)$ such that X is an arbitrary (non-empty) set, $\ast$ is a continuous t-norm and M is a function defined on $X^2\times]0,+\infty[$ with values in $]0,1[$ satisfying the following conditions, for all x, y, z $\in$ X, and s,t > 0 :

(i) $M(x,y,t) > 0$,

(ii) $M(x,y,t) =1$ if and only if $x = y$,

(iii) $M(x,y,t) = M(y,x,t)$,

(iv) $M(x,y,t) \ast M(y,z,s) \le M(x,z,t+s)$,

(v) $M(x,y,\cdot):]0,+\infty[\to [0,1]$ is continuos.

Then M is called a fuzzy metric on X. The function M(x, y, t) denote the degree of nearness between x and y with respect to t, also condition (ii) is equivalent to

$M(x, x, t) = 1$ for all $x \in X$ and $t > 0$, and $M(x, y, t) < 1$ for all $x \neq y$ and $t > 0$.

Remark 1.4 [10] In fuzzy metric space X, M(x, y,.) is non-decreasing for all x, y $\in$ X.

Example1.5 Let (X, d) be a metric space. Denote $a * b = ab$ for all $a, b \in [0, 1]$ and let M_d be a fuzzy set on $X^2 \times]0, \infty[$ defined as follows:

$M_d(x, y, t) = \dfrac{kt^n}{kt^n + md(x,y)}$, for all $k, m, n\in R^+$, $x, y\in X$, Then $(X, M_d,*)$ is a fuzzy metric space.

Remark 1.6 Note the above example holds even with the t-norm $a \ast b = \min \{a, b\}$ and hence M is a fuzzy metric with respect to any continuous t-norm.

In above example by putting $k = m = n = 1$, we get

$$M_d (x, y, t) = \frac{t}{t + d(x,y)}$$

We call this fuzzy metric induced by a metric *d* the standard fuzzy metric.

Definition 1.7[8] Let (X, M, $*$) be a fuzzy metric space and let r $\in$ (0, 1),t > 0 and x$\in$X. The set B(x, r, t) = {y $\in$ X: M(x, y, t) > 1 − r} is called the open ball with center x and radius r with respect to t.

Theorem 1.8[8] Every open ball B(x, r, t) is an open set.

George and Veeramani proved in [8] that every fuzzy metric space (X, M,$*$) on X generates a topology τ_M on X which has as a base the family of open sets of the form

{B_M(x ,r ,t) : x $\in$X; 0 < r <1 ,t > 0} ,they proved that (X,τ_M) is Hausdorff first countable topological space, where

τ_M = {A $\subset$ X :for each x $\in$ X, there exist t > 0, r $\in$(0, 1) such that B(x, r, t) $\subset$ A}.Also if (X,τ) metric space, then the topology induced by d coincides with the topology τ_{M_d} induced by the fuzzy metric M_d.

Example1.9 Let X = **N** (where **N** is the set of natural numbers) and we define

a * b = max{0, a + b −1} for all a, b $\in$ [0, 1] and let M be a fuzzy set on $X^2 \times (0,\infty)$ defined as follows:

$$M(x,y,t)=\begin{cases}\dfrac{x}{y} & ,if \quad x \le y\\ \dfrac{y}{x} & ,if \quad y \le x\end{cases}$$

For all x, y $\in$X, and t > 0 then (X, M, *) is a fuzzy metric space. M induces on X the discrete topology, (in fact, for x ≠ y we have M(x, y, t)$\le \dfrac{x}{(x+1)}$ .Now, if we choose r such that

0 < r < $\dfrac{x}{(x+1)}$, then y $\in$B(x, r, t) if and only if M(x, y, t) > 1 − r > $\dfrac{x}{(x+1)}$ and ,therefore, B(x, r, t) ={x}).

Definition 1.10[9] Let (X,M,$*$) be a fuzzy metric space and let
r $\in$ (0,1),t > 0 and x $\in$ X.

The set B[x, r, t] = {y $\in$ X: M(x, y, t) ≥ 1 − r} is called the closed ball with center x and radius r with respect to t.

Theorem 1.11[9] Every closed ball *B*[*x*, *r*, *t*] is a closed set.

Theorem 1.12 [8] A sequence (x_n) in a fuzzy metric space (X,M, $*$) converges to x if and only if M(x_n, x, t)→1 as n→∞.

Example 1.13 Let X = **R**, the set of all real numbers, and a $*$ b = min {a, b}.

For x; y ∈X; t ≥ 0, define

$$M(x,y,t)=\begin{cases}\dfrac{t}{t+|x-y|} & ,if \quad t>0\\ 0 & ,if \quad t=0\end{cases}$$

Then M is a fuzzy metric on **R.** Let (s_n) be a sequence defined as

$s_n=\frac{1}{n}$, for n∈ N. Then $M(s_n,x,t)\to 1$ as n→∞ .

Definition 1.14 [8] A sequence(x_n) in a fuzzy metric space (X,M,∗) is a Cauchy sequence if and only if for each r∈(0, 1) and each t > 0 there exists n_0 ∈**N** such that

$M(x_n, x_m, t) > 1 - r$ for all n, m ≥ n_0, i.e. $\lim_{n\to\infty} M(x_n,x_m,t)=1$,

for every t > 0.

Definition 1.15 [8] A fuzzy metric space in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

Example 1.16 Let X = $\mathbf{R}^+$, with the metric d defined by d(x,y) = ∣x− y∣, and t-norm a ∗ b = min {a, b},we defined

M (x, y, t) = $\frac{t}{t+d(x,y)}$, for all x, y ∈ X, t > 0.Clearly (X,M,∗) is a complete fuzzy metric spaces.

Theorem 1.17 [9] Let (X,M, ∗) be a fuzzy metric space ,then for each metric d on X compatible with M ,the following hold;

1. A sequence (x_n) in X is Cauchy in (X,M, ∗) if and only if it
 - is Cauchy in (X, d).
2. (X, M,∗) is complete if and only if (X,d) is complete.

Lemma 1.18

Let (X,M,∗) be a fuzzy metric space. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ then $\lim_{n\to\infty} M(x_n,y_n,*) = M(x,y,*)$.

Defnition1.19[8]. A fuzzy metric space (X,M,∗)is called compact if every sequence has a convergent subsequence.

Theorem1.20[8,13] Let (X,M,∗) be a compact fuzzy metric space and let T : X →X be a self-map satisfying:

$$M(Tx, Ty, t) > M(x, y, t)$$

for all x, y ∈ X such that x ≠ y, and t > 0. Then T has a unique fixed point.

Some topological properties in fuzzy metric spaces

In the first of this section, we introduce the following concepts:

The definition of continuity of a mapping f from a fuzzy metric space (X,M) to a fuzzy metric space (Y,N) can be given using four parameters as follows.

Definition2.1 A mapping f from fuzzy metric space (X, M,*) to (Y,N,★) is called continuous at $x_0 \in X$ if given $0 < r < 1$ and $t > 0$ there exists $0 < r_0 < 1$ and $t_0 > 0$ such that $M(x_0,x,s) > 1-r_0$ implies $N(f(x_0),(f(x),t) > 1-r$.

Obviously the condition of continuity of a mapping f between stationary fuzzy metric spaces only needs two parameters. Then, thinking in stationary fuzzy metric spaces and according to the concept of t-uniformly continuous function we give the next definition, by mean of three parameters.

Definition2.2. A mapping f from fuzzy metric space (X, M,*) to (Y,N,★) is called uniformly continuous if for each $0< r < 1$ and each $t > 0$ there exists $0 < r_0 < 1$ (depending on r alone) and $t_0 > 0$ such that $N(f(x),(f(y),t) > 1-r$ whenever $M(x,y,t_0) > 1- r_0$.

It is clear that every uniformly continuous mapping from the fuzzy metric space (X,M,∗) to the fuzzy metric space (Y,N,★) is continuous from (**X,**τ_M**)** to(Y,τ_N).

It is easy to verify that this definition is equivalent to consider$f$: $(X, U_M) \to (Y, U_N)$ as uniform continuous with respect to the uniformities U_M and U_N deduced from M and

N respectively, and then it is continuous from (X, τ_M) to (Y, τ_N).

Similarly to the classical metric case, if $f : (X,M) \to (Y,N)$ is uniformly continuous and (x_n) is a Cauchy sequence in X then $\{f (x_n)\}$ is a Cauchy sequence in Y .

Definition 2.3 Let (X, M,*) be a fuzzy metric space. M is said to be continuous on $X^2 \times]0,\infty[$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x,y,t)$$

Definition2.4 Let (X, M,*) and (Y,N,★) be two fuzzy metric spaces .A mapping f from X to Y is called <u>isometry</u> if for each x, y ∈ X and each t > 0 $M(x,y,t) = N(f(x),(f(y),t)$ and, in this case, if f is a bijection, X and Y are called isometric.

A fuzzy metric completion of (X,M) is a complete fuzzy metric space (X^*,M^*) such that (X,M) is isometric to a dense subspace of X^*. X is called completable if it admits a fuzzy metric completion..

Theorem 2.5 (uniform continuity theorem). Let f be a continuous mapping of compact fuzzy metric space (X, M,*) to fuzzy metric space, (Y,N,★) then f is uniformly continuous.

Proof : Let 0 < s < 1and t > 0 be given, then we can find 0 < r < 1 such that(1–r)*(1–r) > 1–s. Since f: X→ Y is continuous**,** for each x∈ X we can find 0 < r_x < 1 and t_x > 0 such that M(x,y,t_x) > 1–r_x implies N(f(x),(f(y), $\frac{t}{2}$) > 1–r, but 0 < r_x < 1 then we can find s_x < r_x such that(1– s_x) * (1– s_x) > 1– r_x. Since X is compact and $\left\{B(x, s_x, \frac{t_x}{2}) : x \in X\right\}$ is an open covering of X ,there exist $x_1,x_2,\ldots,x_k$ in X such that $X = \bigcup_{i=1}^{k} B\left(x_i, s_{x_i}, \frac{t_{x_i}}{2}\right)$ .put $s_0 = \min\{s_{x_i}\}$ and $t_0 = \min\{\frac{t_{x_i}}{2}\}$,i = 1,2,…,k. For any

x, y∈X if M(x,y,t_0) > 1–s_0,then $M(x, y, \frac{t_{x_i}}{2}) > 1 - s_{x_i}$ .Since x∈ X, there exist a x_i such that $M(x, x_i, \frac{t_{x_i}}{2}) > 1 - s_{x_i}$ **,** hence we have N(f(x),(f(x_i), $\frac{t}{2}$) > 1–r. Now

$$M(y, x_i, t_{x_i}) \geq M(x, y, \frac{t_{x_i}}{2}) * M(x, x_i \frac{t_{x_i}}{2}) \geq 1 - s_{x_i} * 1 - s_{x_i} > 1 - r_{x_i}.$$

Therefore N(f(x),(f(x_i), $\frac{t}{2}$) > 1–r. Now we have

$$N(f(x), f(y), t) \geq N(f(x), f(x_i), \frac{t}{2}) * N(f(y), f(x_i), \frac{t}{2}) \geq (1-r)*(1-r) > 1-s.$$ Hence

f is uniformly continuous. ■

Now we give the definition of t- uniformly continuous mapping**.**

Definition2.6[13] (X,M,∗) be a fuzzy metric space We will say the mapping f : X → X is t-uniformly continuous if for each 0 < r < 1 there exists 0 < r_0 < 1 such that M(x,y,t) > 1–r_0 implies M(f(x),(f(y),t) > 1–r. for each x, y ∈ X and t > 0

Clearly if f is t-uniformly continuous it is uniformly continuous for the uniformity generated by M, and then continuous for the topology deduced from M

The proofs of the following propositions are Straightforward and are omitted.

Proposition 2.7 [13] Let (X,M,∗) be a fuzzy metric space and f : X → X a mapping. Then $f$is t-uniformly continuous iff for each δ> 0 there exists η > 0 such that

$\frac{1}{M(x,y,t)-1} \leq \eta$ implies

$$\frac{1}{M(f(x), f(y),t)} - 1 \leq \delta$$, for each x, y $\in$ X and t > 0.

Definition 2.8[13] Let (X, M, ∗) be a fuzzy metric space .We will say the mapping *f* : X → X is fuzzy contractive if there exists k $\in$]0, 1[such that

$$\frac{1}{M(f(x), f(y),t)} - 1 \leq k(\frac{1}{M(x,y,t)} - 1)$$

Or equivalent

$$M(f(x), f(y),t) \geq \frac{M(x,y,t)}{M(x,y,t) + k(1 - M(x,y,t)}$$ **,**for each x, y $\in$ X and

t > 0. (where K is the contractive constant of *f*)**.**

Theorem 2.9 Let (X, d) be a metric space and let (X, M, ∗) be a fuzzy metric space satisfying:

M (x, y, t) = $\frac{t}{t + d(x,y)}$, for all x, y ∈ X, t ∈ (0, 1]

Then a map *f*: X → X is fuzzy contractive if and only if it is a contractive map on the metric space (X, d).

Proof. Let *f* : X → X and k∈(0, 1). We have for all t ∈(0, 1]

$$M(f(x), f(y),t) \geq \frac{M(x,y,t)}{M(x,y,t) + k(1 - M(x,y,t)},$$

then

$$\frac{M(x,y,t)}{M(x,y,t) + k(1 - M(x,y,t)} = \frac{t}{t + kd(x,y)}$$

And

$$M(f(x), f(y),t) = \frac{t}{t + d(f(x), f(y))}$$

Therefore

d(f(x),f(y)) ≤ kd(x, y)

$$\Leftrightarrow \frac{M(x,y,t)}{M(x,y,t) + k(1 - M(x,y,t)} \leq M(f(x), f(y))$$,for all x,y ∈X.

This shows that *f* is an fuzzy contractive map on(X, M, ∗) if and only if it is a contractive map on the metric space
(X, d). ■

Proposition 2.10[13] Let (X, M, ∗) be a fuzzy metric space. If $f : X \to X$ is fuzzy contractive then f is t-uniformly continuous.

The above definition is justified by the following Proposition :

Proposition 2.11[13] Let (X,d) be a metric space. The mapping $f : X \to X$ is contractive (a contraction) on the metric space (X,d) with contractive constant k iff f is fuzzy contractive, with contractive constant k,on the standard fuzzy metric space $(X, M_d, *)$ induced by d.

Definition2.12 Let X be any nonempty set and (X,M, *) be an fuzzy metric space. Then a sequence $\{f_n\}$ of functions from X to Y is said to be convergence uniformly to a function f from X to Y if for given $0 < r < 1$,$t > 0$,there exists $n_0 \in \mathbf{N}$ such that $M(f_n(x), f(x), t) > 1 - r$ for all $n \geq n_0$ and for all $x \in X$.

Definition2.13. A family **F** of functions from a fuzzy metric space X to a complete fuzzy metric space Y is said to be equicontinuous if for given $0 < r < 1$,$t > 0$, there exist $0 < r_0 < 1$,$t_0 > 0$ such that $M(x, y, t_0) > 1 - r_0 \Rightarrow M(f(x), f(y), t) > 1 - r$ for all $f \in \mathbf{F}$ **.**

Lemma 2.14 Let $\{f_n\}$ be an equicontinuous sequence of functions from an fuzzy metric space X to a complete fuzzy metric space Y. If $\{f_n\}$ converges for each point of a dense subset D of X, then $\{f_n\}$ converges for each point of X and the limit function is continuous.

Proof Let $0 < s < 1$, and $t > 0$ be given. Then we can find $0 < r < 1$, such that $(1 - r) * (1 - r) * (1 - r) > 1 - s$. Since $\mathbf{F} = \{f_n\}$ is equicontinuous family, for given $0 < r < 1$, and $t > 0$, there exist $0 < r_1 < 1$,and $t_1 > 0$ such that for each x, y ∈ X, $M(x, y, t_1) > 1 - r_1 \Rightarrow M(f_n(x), f_n(y), \frac{t}{3}) > 1 - r$ for all $f_n \in \mathbf{F}$ **.** Since D is dense in X, there exists $y \in B(x, r_1, t_1) \cap D$ and $\{f_n(y)\}$converges for that y. Since $\{f_n(y)\}$ is a Cauchy sequence, for given $0 < r < 1$,$t > 0$, there exists $n_0 \in N$ such that $M(f_n(x), f_m(y), \frac{t}{3}) > 1 - r$, for all $m, n \geq n_0$. Now for any x ∈ X, we have

$$M(f_n(x), f_m(y), t)$$

$$\geq M(f_n(x), f_n(y), \tfrac{t}{3}) * M(f_n(y), f_m(y), \tfrac{t}{3}) * M(f_m(x), f_m(y), \tfrac{t}{3})$$

$$\geq (1 - r) * (1 - r) * (1 - r) \geq 1 - s .$$

Hence $\{f_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, $f_n(x)$ converges.

Let $f(x) = \lim f_n(x)$. We claim that f is continuous. Let $0 < s_0 < 1$, and $t_0 > 0$ be given. Then we can find $0 < r_0 < 1$, such that

$(1-r_0) * (1-r_0) * (1-r_0) \geq 1-s_0$. Since **F** is equicontinuous, for given $0 < r_0 < 1$ and $t_0 > 0$, there exist $0 < r_2 < 1$and $t_2 > 0$ such that $M(x, y,t_2)>1- r_2 \Rightarrow M(f_n(x), f_n(y), \frac{t_0}{3}) > 1-r_0$ for all $f_n \in$**F**.

Since $f_n(x)$ converges to $f(x)$, for given $0 < r_0 < 1$ and $t_0 > 0$, there exists $n_1 \in N$ such that $M(f_n(x), f_n(x), \frac{t_0}{3}) > 1-r_0$.

Also since $f_n(y)$ converges to $f(y)$, for given $0 < r_0 < 1$ and $t_0 > 0$, there exists $n_2 \in N$ such that $M(f_n(y), f_n(y), \frac{t_0}{3}) > 1-r_0$ for all $n \geq n_2$. Now for all $n \geq \max\{n_1,n_2\}$, we have

$M(f(x),f(y),t_0)$

$\geq M(f(x),f_n(x),\frac{t_0}{3}) * M(f_n(x),f_n(y),\frac{t_0}{3}) * M(f_n(y), f(y),\frac{t_0}{3})$

$\geq (1-r_0) * (1-r_0) * (1-r_0) \geq 1-s_0$.

Hence f is continuous. ■

Product of Fuzzy Metric Spaces

In [16] R. Mohd, and S. Mohd are introduced the definition of product two fuzzy metric spaces in the sense of Egbert [6] as follows:

Definition 3.1[16] Let $(X, M_X,*)$ and $(Y, M_Y,*)$ are two fuzzy metric spaces defined with same continuous t-norms *.Let ◊ be a continuous t-norm. The ◊-product of $(X, M_X ,*)$ and $(Y, M_Y ,*)$ is the product space $(X \times Y, M_\Diamond , *)$ where $X \times Y$ is the Cartesian product of the sets X and Y , and $M_\Diamond$ is the mapping from $(X \times Y \times (0, 1)) \times (X \times Y \times (0, 1))$ into [0, 1] given by

$M_\Diamond (p, q, t + s) = M_1(x_1,x_2,t) \Diamond M_2(y_1,y_2,s)$ …………(1)

for every $p = (x_1,y_1)$ and $q =(x_2,y_2)$ in $X \times Y$ and $t + s \in (0,1)$.

As an immediate consequence of Definition 3.1, we have from [16]

Theorem 3.2. [16] If $(X, M_X ,*)$ and $(Y, M_Y ,*)$ are fuzzy metric spaces under the same continuous t-norm *, then their * -product $(X \times Y, M_* , *)$ is a fuzzy metric space under * .

Example 3.3. Let (X, d_X) and (Y, d_Y) are metric spaces and $(X \times Y, d)$ be their product with

$d(p, q) = \text{Max}\{d_X (x_1,x_2),d_Y (y_1,y_2)\}$for each $p = (x_1,y_1)$and

$q = (x_2,y_2)$ in $X \times Y$.

Define $a \Diamond b = \text{Min}\{a, b\}$ for all $a, b \in [0, 1]$ and let $M_d(p,q,t) = \frac{1}{t + d(p,q)}$.

Then $(X \times Y, M_d,*)$ is a ◊ - product of (X, d_X) and (Y, d_Y).

Proof It is suffices to prove the condition (1).To this end

$$M_d(p,q,t) = \frac{t}{1+d(p,q)} = \frac{t}{t + Max\{d_X(x_1,x_2), d_Y(y_1,y_2)\}}$$
$$= \frac{t}{Max\{t + d_X(x_1,x_2), d_Y(y_1,y_2)\}}$$
$$= Min(\frac{t}{t+d_X(x_1,x_2)}, \frac{t}{t+d_Y(y_1,y_2)})$$
$$= (\frac{t}{t+d_X(x_1,x_2)})\Diamond(\frac{t}{t+d_Y(y_1,y_2)}).$$

Whence, $M_d(p,q,t) = \boldsymbol{M}_{d_X} \Diamond \boldsymbol{M}_{d_Y}$. ■

Definition3.4 [16] Let ◊ and * are continuous t-norms. We say that ◊ stronger than *, if for each $a_1, a_2, b_1, b_2 \in [0, 1]$,

$(a_1 * b_1) \Diamond (a_2 * b_2) \geq (a_1 \Diamond a_2) * (b_1 \Diamond b_2)$.

Lemma 3.5 If ◊ is stronger than * then ◊ ≥ *

Proof: From Definition 3.4, by setting $a_2 = b_1 = 1$, so, get

$a_1 \Diamond b_2 \geq a_1 * b_2$,

i.e.,◊ ≥ *. ■

Theorem3.6[16] Let (X, M_X,*) and (Y, M_Y,*) are two fuzzy metric spaces defined with same continuous t-norms *. If there exists a continuous t-norm ◊ stronger than *, then the ◊ -product (X × Y, $M_\Diamond$,*) is a fuzzy metric space under *.

Proof: The axioms (i, ii, and iii,) of definition (1.3) are obvious, it is suffices to prove axiom (V) and (iv) .

Let $p = (x_1,y_1)$,$q = (x_2,y_2)$,$r = (x_3,y_3)$ are in X × Y . Then

$M(p, r, 2\lambda) = (M_X(x_1,x_3, \lambda) \Diamond M_Y(y_1,y_3, \lambda))$

$\geq(M_X(x_1,x_2,\lambda/2)*M_X(x_2,x_3,\lambda/2))\Diamond(M_Y(y_1,y_2,\lambda/2)*M_Y(y_2,y_3,\lambda/$**)**

$\geq(M_X(x_1,x_2,\lambda/2)\Diamond M_X(x_2,x_3,\lambda/2))*(M_Y(y_1,y_2,\lambda/2)\Diamond M_Y(y_2,y_3,\lambda/2)$

$=M_\Diamond(p, q,\lambda) * M_\Diamond(q, r,\lambda)$.

The continuity of the t-norms implies the function

$M_\Diamond(p, q, \cdot): (0, \infty) \to [0, 1]$ is continuous. ■

Corollary 3.7 If (X, M_X,$*_1$) and (Y, M_Y,$*_2$) are fuzzy metric spaces and if there exists a continuous t-norm stronger than $*_1$and $*_2$ then their ◊ product is a fuzzy metric space under ◊.

We now turn to the question of topologies in the -product spaces and give the following result:

Theorem 3.8 Let $(X_1,M_1,*)$ and $(X_2,M_2,*)$ be fuzzy metric spaces under the same continuous t-norm * .Let U denote the neighborhood system in$(X_1 \times X_2,M_*, *)$ and let V denote the neighborhood system in $(X_1 \times X_2,M_*, *)$ consisting of the Cartesian products $B(x_1, r, t) \times B(x_2\ r, t)$ where $x_1 \in X_1$, $x_2 \in X_2$, $r \in (0, 1)$ and $t > 0$. Then U and V induce the same fuzzy topology on$(X_1 \times X_2,M_*, *)$.

Proof: Clearly, since * is continuous, U and V are bases for their respective topology. So, it is suffices to prove that for each $V^` \in V$ there exists a $U^` \in U$ such that $U \subseteq V$, and conversely. Let $A_1 \times A_2 \in V$. Then there exist neighborhoods $B(x_1,r, t)$ and $B(x_2,r, t)$ contained in A_1 and A_2 respectively. Let $r = Min\{r_1,r_2\}$, $t = Min\{t_1,t_2\}$, and let $x = (x_1,x_2)$. Here, we shall show that $B(x_1,r, t) \in A_1 \times A_2$. Let $y = (y_1,y_2) \in B(x,r, t)$, then we have

$M_1(x_1,y_1,t_1) = M_1(x_1,y_1,t_1) * 1 \geq M_1(x_1,y_1,t_1) * M_1(x_2,y_2,t_2)$

$\geq M_1(x_1,y_1,t) \geq M_1(x_2,y_2,t)$

$= M(x, y, t) > 1- r \geq 1 - r_1$.

Similarly, we can show that $M_2(x_2,y_2,t_2) > 1 - r_2$. Thus $y_1 \in B(x_1, r_1,t_1)$ and $y_2 \in B(x_2,r_2,t_2)$ which implies that $B(x,r,t) \in A_1 \times A_2$.

Conversely, suppose that $B(x,r, t) \in U$. Since *is continuous, there exists an $\sigma \in (0, 1)$ such that

$(1 - \sigma) * (1 - \sigma) > 1 - r$.

Let $y = (y_1,y_2) \in B(x_1, \sigma, t) \times B(x_2,\sigma, t)$. Then

$M(x, y, t) = M_1(x_1,y_1,t) * M_2(x_2,y_2,t) \geq (1 -\sigma) * (1 -\sigma) > 1 - r$

so that $y \in B(x,r, t)$ and $B(x_1,r, t) \times B(x_2, r, t) \subseteq B(x,r, t)$. This completes the proof. ■

Definition3.9 Let (X, M,*) be a fuzzy metric space.A mapping $f : X \to X$ is said to be fuzzy **contraction** if there exists a $k \in (0, 1)$ such that

$M(f(x),f(y),t) \geq M(x, y, t/k)$ for all $x, y \in X$

Theorem 3.10 [9] Let (X, M, *) be a complete fuzzy metric space such that $\lim_{t\to\infty} M(x, y, t) = 1$for all $x, y \in X$.

Let $f: X \to X$ be a contractive mapping. Then f has a unique fixed point.

We use now concept of convergence uniformly with contraction for fixed point

Theorem 3.11 Let (X, M, *) be a fuzzy metric space with $a*b = Min \{a,b\}$.Let $f_n: X \to X$ be a mapping with at least one fixed point x_n for each n =1, 2, ⋯,and $f : X \to X$ be a fuzzy contraction mapping with fixed point x_0.

If the sequence (f_n) converges uniformly to f, then the sequence (x_n) converges to x.

Proof: Let $k \in (0, 1)$ and choose a positive number $n_0 \in$ such that

$n \geq n_0$ implies

$$M(f_n(x), f(x), (1-k)t) > 1 - r$$

where $r \in (0, 1)$ and $x \in X$. Then, if $n \geq n_0$, we have

$M(x_n, x, t) = M(f_n(x_n), f(x), t)$

$\geq M(f_n(x_n), f(x_n), (1-k)t) * M(f_n(x), f(x), kt)$

$> Min(1-r, M(x_n, x, t))$.

Hence, $M(x_n, x, t) \to 1$ as $n \to \infty$. This proves that (x_n) converges to x. ■

In [8, Theorem 8], the author gives the following Edelstein contraction theorem: "Let (X, M, *) be a compact fuzzy metric space.

Let T: X →X be a mapping satisfying

$$M(T(x),T(y), \cdot) \geq M(x, y, \cdot) \text{ for all } x \neq y.$$

Then, T has a unique fixed point.

Now, if T : X →X is a fuzzy contractive mapping satisfying, for some $k \in]0, 1[$:

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right) \quad ,t > 0$$

then $M(T(x),T(y),t) \geq M(x, y, t), x \neq y$ and thus, the mentioned Edelstein contraction theorem is satisfied for fuzzy contractive mappings.

Now, we prove the following theorem

Theorem 3.12 let f be a mapping of a compact fuzzy metric space (X,M,*) into itself and let for some positive integer n, f^n be fuzzy contractive. Then f has a unique fixed point.

Proof If the contractive map f^n moves all the point of X then by compactness there exists a point x such that $M(f^n(x),x,t)$ is minimal.

But $M(f^n(x),f^n(f^n(x)),t) > M(x,f^n(x),t)$,which contradicts the minimality of $M(x, f^n(x),t)$.Therefore, we must have

$M(x, f^n(x), t) = 1$ if and only if $f^n(x) = x$. To show uniqueness, assume $f^n(x) = y$ for some $y \in X$, then for $t > 0$ we have

$1 \geq M(x, y, t) = M(f^n(x), f^n(y),t) \geq M(x,y,t)$ implying there by that x is a unique fixed point of f^n, then x is a fixed point of f is as follows:

$$x = f^n(x) \Rightarrow f(x) = f(f^n(x)$$

$$= f^n(f(x)$$

$\Rightarrow$ x = f(x).

Lastly, uniqueness of x as a fixed point of f may be easily verified. ■

Remark 3.13: The ideas used in above proof may also be used to give a different proof of the contraction mapping principle. Which proceeds as follows (recall the hypotheses of the contraction mapping principle).

For any x $\in$ M and any natural number n with using induction we have

$$1 \geq M(x, y, t) = M(f^n(x), f^n(y), \frac{t}{k}) \geq M(x, y, \frac{t}{k}) \geq M(x, y, \frac{t}{k^2}) \geq \ldots \geq M(x, y, \frac{t}{k^n}).$$

Clearly $(\frac{t}{k^n})$ is an increasing sequence, then by assumption and clearly, $\lim_{n\to\infty} M(x, y, \frac{t}{k^n}) = 1$ hence M (x, y, t) = 1and x = y

Theorem 3.14 Let f^n be a fuzzy contractive mappings in each variable separately on compact fuzzy metric space (X × Y, $M_\Diamond$, *) into itself. Then f has unique fixed point.

Proof. For every p = (x_1,y_1) and q = (x_2,y_2) in X × Y and

t + s $\in$ (0,1) we have

$$M(p, q, t + s) = M_1(x_1,x_2,t) \Diamond M_2(y_1,y_2,s)$$

$$= (M_1(x_1,x_2,t)\Diamond 1)*(1\Diamond M_2(y_1,y_2,s))$$

$$= (M_1(x_1,x_2,\tfrac{t}{2}) \Diamond M_2(y_1,y_1,\tfrac{s}{2}))* (M_1(x_2,x_2,\tfrac{t}{2}) \Diamond M_2(y_1,y_2,\tfrac{s}{2}))$$

$$= M((x_1,y_1),(x_2,y_1), \tfrac{t}{2}+\tfrac{s}{2}) * M((x_2,y_1),(x_2,y_2), \tfrac{t}{2}+\tfrac{s}{2})$$

$$< M(f^n(x_1,y_1),f^n(x_2,y_1), \tfrac{t}{2}+\tfrac{s}{2})* M(f^n(x_2,y_1),f^n(x_2,y_2), \tfrac{t}{2}+\tfrac{s}{2})$$

$$\leq M(f^n(x_1,y_1),f^n(x_2,y_2),t + s).$$

Therefore f^n is contractive and by Theorem **(3.12)** has unique fixed point ■

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