HAMILTON-JACOBI TREATMENT OF LAGRANGIANS WITHIN FRACTIONAL CAPUTO DERIVATIVES

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Abstract

In this work, the Hamilton-Jacobi formulation of fractional Caputo Lagrangians of linear velocities is investigated. The fractional Hamilton-Jacobi equations of motion for several potential systems are derived. Under certain conditions on the potential, it is shown that the action integral is independent on the fractional Caputo derivatives.

Keywords: Fractional Caputo Derivatives, Lagrangians of Linear velocities, The fractional Hamilton-Jacobi equation

Introduction

The classical Hamilton-Jacobi equation represents a reformulation of classical mechanics which is equivalent to other formulations such as Newton's laws of motion, Lagrangian mechanics and Hamiltonian mechanics. In addition, the Hamilton-Jacobi equation is useful in finding the conserved quantities for mechanical systems, which may be possible even when the mechanical problem itself cannot be solved completely [1]. The Hamilton-Jacobi theory represents the only formulation of mechanics in which the motion of a particle can be represented as a wave.

On the other hand, the investigation of the fractional Hamilton-Jacobi equation is still at the beginning of its development. Fractional calculus deals with the generalization of differentiation and integration to non-integer orders [2-6]. For these reasons, a large body of mathematical knowledge on fractional integrals and derivatives has been constructed.

More contributions and interesting applications of fractional calculus can be found in the references [7-13]. For examples, Riewe has used the fractional calculus to develop a formalism which can be used for both conservative and non conservative systems. Tarasov *et al* considered the fractional generalization of nonholonomic constraints defined by equations with fractional derivatives. They proved that fractional constraints can be used to describe the evolution of dynamical systems in which some coordinates and velocities are related to velocities through a power-law memory function.

Recently, the fractional constrained Lagrangian and Hamiltonian were analyzed [14-16]. The notion of the fractional Hessian was introduced and the Euler-Lagrange equations were obtained for a Lagrangian linear in velocities.

Quantization of systems with fractional derivatives is a novel area in the theory of application of fractional differential and integral calculus. The path integral quantization of fractional mechanical systems with constraints is discussed in [17]. Schrödinger equation was considered with the first order time derivative modified to fractional Caputo ones in [18]. Moreover, Laskin studied some properties of the fractional Schrödinger equation. He proved the Hermiticity of the fractional Hamilton operator and established the parity conservation law for fractional quantum mechanics. As physical applications of the fractional Schrödinger equation he found the energy spectra of a hydrogenlike atom (fractional “Bohr atom”) and of a fractional oscillator in the semiclassical approximation [19].

The aim of this study is to extend the Agrawal’s approach to classical fields with fractional derivatives.

The plan of this paper is as follows:

In Sec.2, the fractional Hamilton-Jacobi differential equations are investigated in terms of fractional Caputo derivatives. In Sec.3, three examples are studied. Finally, Sec.4 is dedicated to our conclusions.

Fractional Hamilton-Jacobi Formulation

Consider the following fractional Caputo Lagrangian with n generalized coordinates:

$$L = b_i(q)\,{}_a^C D_t^\alpha q - V(q) \qquad i = 1,2,3....,n\,. \tag{1}$$

The generalized fractional Caputo momenta corresponding to this Lagrangian are:

$$p_i = \frac{\partial L}{\partial\,{}_a^C D_t^\alpha q} = b_i \equiv -H_i\,. \tag{2}$$

Following [20,24], these equations represent constraint equations:

$$H_i' = p_i - b_i(q)\,. \tag{3}$$

The Hamiltonian H_0 reads

$$H_0 = p_i\,{}_a^C D_t^\alpha q - L = V\,.$$

The corresponding fractional Caputo Hamilton-Jacobi partial differential equations (HJPDEs) are

$$H_0' = p_0 + H_0 = {}_a^C D_t^\alpha S + V = 0 \,;$$

$$H_i' = p_i + H_i = {}_a^C D_{q_i}^\alpha S - b_i = 0 \,.$$

The total derivative of the fractional Caputo Hamilton-Jacobi function can be obtained as:

$$dS(q,t) = {}_a^C D_{q_i}^\alpha S \, dq_i + {}_a^C D_t^\alpha S \, dt \,. \tag{4}$$

Using the above fractional Caputo HJPDEs, we get

$$dS = b_i \, dq_i - V dt \,, \tag{5}$$

which can be integrated to give

$$S = \int b_i \, dq_i - \int V \, dt \,. \tag{6}$$

Now, using the fact hat

$$\int d(b_i \, q_i) = b_i q_i = \int b_i dq_i + \int q_i \, db_i \,, \tag{7}$$

the above fractional Caputo Hamilton- Jacobi function Eq.(6) reduces to

$$S = \frac{1}{2}\left[b_i q_i + \int b_i \, dq_i - \int q_i \, db_i\right] - \int V \, dt \,. \tag{8}$$

After some rearrangements, Eq.(8) becomes

$$S = \frac{1}{2} b_i q_i - \frac{1}{2}\int \left[q_i db_i - b_i dq_i + 2V dt\right]. \tag{9}$$

Assuming that the functions b_i and V(q) satisfy the following conditions

$$({}_a^C D_{q_j}^\alpha b_i) q_j = b_i \,, \quad ({}_a^C D_{q_j}^\alpha V) q_j = 2V \,, \tag{10}$$

Eq.(9) then becomes

$$S = \frac{1}{2} b_i q_i - \frac{1}{2}\int q_j [db_j - ({}_a^C D_{q_j}^\alpha b_i) dq_i + ({}_a^C D_{q_j}^\alpha V) dt]. \tag{11}$$

However, in order that S is an integrable function, the terms inside the brackets must be zero, i.e.

$$db_j - ({}_a^C D_{q_j}^\alpha b_i) dq_i + ({}_a^C D_{q_j}^\alpha V) dt = 0 \,.$$

(12)

In fact, this equation represents the equation of motion for the coordinate q_j.

Accordingly, Eq.(11) reduces to

$$S = \frac{1}{2} b_i q_i + c \,. \tag{13}$$

Applications

The fractional Caputo Lagrangian of a two dimensional quadratic potential of the form v = $\frac{1}{2}\left(q_1^2 + q_2^2\right)$ is tested as shown below.

$$L = q_2 \; {}_a^C D_t^\alpha q_1 - q_1 \; {}_a^C D_t^\alpha q_2 - \frac{1}{2}\left(q_1^2 + q_2^2\right), \tag{14}$$

Here the functions b_1 and b_2 read

$$b_1 = q_2 \qquad b_2 = -q_1$$

Using Eq.(12), the fractional equations of motion for q_1 and q_2 are respectively

$$dq_2 + ({}_a^C D_{q_1}^\alpha q_1) dq_2 + \frac{1}{2} {}_a^C D_{q_1}^\alpha (q_1)^2 \, dt = 0 \,, \tag{15}$$

$$-dq_1 - ({}_a^C D_{q_2}^\alpha q_2) dq_1 + \frac{1}{2} {}_a^C D_{q_2}^\alpha (q_2)^2 \, dt = 0 \,. \tag{16}$$

The left Caputo fractional derivative is defined as [6]

$${}_a^C D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau) d\tau \,.$$

Applying this definition for $f(x) = q_1$ and $f(x) = q_2$, we get

$${}_a^C D_{q_1}^\alpha q_1 = \frac{1}{\Gamma(2-\alpha)} (q_1 - a)^{1-\alpha} \,,$$

$${}_a^C D_{q_2}^\alpha q_2 = \frac{1}{\Gamma(2-\alpha)} (q_2 - a)^{1-\alpha} \,.$$

Substituting these equations in equations (15) and (16), we obtain

$$dq_2 + \frac{1}{\Gamma(2-\alpha)} (q_1 - a)^{1-\alpha} dq_2 + \frac{1}{\Gamma(3-\alpha)} (q_1 - a)^{1-\alpha} [q_1 + a(1-\alpha)] dt = 0 \,; \tag{17}$$

$$-dq_1 - \frac{1}{\Gamma(2-\alpha)} (q_2 - a)^{1-\alpha} dq_1 + \frac{1}{\Gamma(3-\alpha)} (q_2 - a)^{1-\alpha} [q_2 + a(1-\alpha)] dt = 0 \,. \tag{18}$$

As a special case if $\alpha = 1$, we get

$$2\dot{q}_2 + q_1 = 0 \,; \quad 2\dot{q}_1 - q_2 = 0 \,. \tag{19}$$

From Eq.(13), the action integral is

$$S = c .$$

Which means that this action is independent on the fractional Caputo derevatives

As a second application, the fractional Caputo Lagrangian is considered for a three dimensional potential of the form v = $2q_1 q_3 - \frac{1}{2} q_3^2$:

$$L = q_1 \, {}_a^C D_t^\alpha q_1 + q_2 \, {}_a^C D_t^\alpha q_2 + q_3 \, {}_a^C D_t^\alpha q_1 - q_1 \, {}_a^C D_t^\alpha q_3 - 2q_1 q_3 + \frac{1}{2} q_3^2 , \tag{20}$$

The functions b_i (i= 1,2,3) are

$$b_1 = q_1 + q_3 , \quad b_2 = q_2 , \quad b_3 = -q_1 . \tag{21}$$

Using Eq.(12), the equations of motion for q_1, q_2 and q_3 are respectively

$$d(q_1 + q_3) - ({}_a^C D_{q_1}^\alpha q_1) dq_1 + ({}_a^C D_{q_1}^\alpha q_1) dq_3 + 2({}_a^C D_{q_1}^\alpha q_1) q_3 dt = 0 ; \tag{22}$$

$$dq_2 - ({}_a^C D_{q_2}^\alpha q_2) dq_2 = 0 ; \tag{23}$$

$$-dq_1 - ({}_a^C D_{q_3}^\alpha q_3) dq_1 + 2({}_a^C D_{q_3}^\alpha q_3) q_1 dt = 0 . \tag{24}$$

As α goes to one, we obtain

$$\dot{q}_3 + q_3 = 0 ; \qquad 2\dot{q}_1 - 2q_1 + q_3 = 0 .$$

Eq.(13) gives the action integral as

$$S = \frac{1}{2}\left(q_1^2 + q_2^2\right) + c .$$

In the third case the four dimensional fractional Caputo Lagrangian for the arbitrary potential v = $-\frac{1}{2}\left(q_4^2 - 2q_2 q_3 - q_3^2\right)$ is applied.

$$L = (q_2 + q_3) \, {}_a^C D_t^\alpha q_1 + q_4 \, {}_a^C D_t^\alpha q_3 + \frac{1}{2}\left(q_4^2 - 2q_2 q_3 - q_3^2\right), \tag{25}$$

The functions b_i (i=1,2,3,4) read

$$b_1 = q_2 + q_3 ; \quad b_2 = 0 ; \quad b_3 = q_4 , \; b_4 = 0 . \tag{26}$$

Using Eq.(12), the equations of motion for q_1, q_2, q_3 and q_4 are

$$d(q_2 + q_3) = 0 ; \tag{27}$$

$$-({}_a^C D_{q_2}^\alpha q_2) dq_2 + q_3 ({}_a^C D_{q_2}^\alpha q_2) dt = 0 ; \tag{28}$$

$$dq_4 - ({}_a^C D_{q_3}^\alpha q_3) dq_1 + {}_a^C D_{q_3}^\alpha [q_2 q_3 + \frac{1}{2}(q_3)^2] dt = 0 ; \tag{29}$$

$$-({}_a^C D_{q_4}^{\alpha} q_4) dq_3 + {}_a^C D_{q_4}^{\alpha} (-\frac{1}{2} q_4^2) dt = 0 . \qquad (30)$$

If α = 1, these equations reduce to

$$\dot{q}_2 + \dot{q}_3 = 0; \quad \dot{q}_1 - q_3 = 0; \quad \dot{q}_4 - \dot{q}_1 + q_2 + q_3 = 0; \quad \dot{q}_3 + q_4 = 0. \ (31)$$

The action integral is

$$S = \frac{1}{2}\left(q_1 q_2 + q_1 q_3 + q_3 q_4\right) + c .$$

Despite the fact that the potential is three dimensional, the action S is four dimensional, but independent of the Caputo derivatives.

Conclusion

We studied the formulation of the Hamilton-Jacobi function with fractional Caputo derivatives for Lagrangians with linear velocities. The fractional Hamilton-Jacobi equations of motion for several Lagrangian systems with different potentials were investigated. It is shown that in the limit α goes to 1, the equations of motion are in agreement with the ordinary Hamilton-Jacobi equations of motion. Besides, The action integrals did not show an explicit dependence on fractional Caputo derivatives.

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