

The Stress-Strength Model: $R=P(Y<X)$ Gamma Case by Mathematica

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Doi: 10.19044/esj.2019.v15n9p177 [URL:http://dx.doi.org/10.19044/esj.2019.v15n9p177](http://dx.doi.org/10.19044/esj.2019.v15n9p177)

Abstract

In this paper, we considered the estimation of $R=P(Y<X)$, dubbed as Stress-Strength Model (SSM), in a framework of gamma distribution with unknown scale and shape parameters. This is not an easy computing problem in conventional mathematics. For such problems, a very important option is using a Mathematica software, which has been employed to obtain solutions to this problem.

Keywords: Gamma distribution, Stress-Strength model, Maximum likelihood estimation, Monte-Carlo Simulation, Kolmogorov-Smirnov one sample test

Introduction

Every physical component or system possesses an inherent strength against applied stress. The quantity $R=P(Y<X)$ is often used in mechanical reliability theory, and it is termed as the Stress-Strength Model (SSM) (Johnson, 1988; Kotz & Pensky, 2003). Stress-Strength reliability is of interest in many fields such as economics, engineering, biology, medicine, agriculture, computers, etc. (Tamura, 2002; Koga & Aoyama, 2005; Ghosh et al., 2011; Otsuka et al., 2011; Otsuka et al., 2012; Stojković et al., 2017; Popov et al., 2018; Kobayashi et al. 2018). In most studies, it is accepted that X and Y are continuous, independent, and have the same univariate family of distributions. The explicit expressions for R and its properties have been derived for majority of useful distributions, e.g. normal, uniform, exponential, gamma, Pareto, Weibull, etc. (Kotz & Pensky 2003; Min & Sun, 2013).

Constantine and Karson (1986) and Ismail et al. (1986) studied R when X and Y are from gamma distributions with known shape parameters. However, limited results are available in the case where X and Y are independent gamma variable.

The two parameters gamma distribution is a computationally complex distribution; its distribution function, inverse distribution function, and maximum likelihood estimators (MLEs) are not in closed forms. This is the reason why conventional arithmetic computations are tedious. Recently, several user-friendly softwares are available to handle these types of problems. Mathematica (Wolfram, 2000) is one of the leading software among them.

In this paper, we explored the estimation of $R=P(Y<X)$ in the framework of gamma distribution with unknown scale and shape parameters using Mathematica version 10.2.

The two-parameter gamma distribution represents the scale (β) and shape (α) parameters. As a result of these parameters, it has become a flexible model (from skew exponential to quasi symmetrical) in many physical situations. Unfortunately, some of its properties are mathematically intractable which make this distribution less popular in modelling as mentioned above. The quantity $R=P(Y<X)$ cannot be computed in closed form for arbitrary shape parameter (α).

The Stress-Strength Model

The Stress-Strength Model has been widely used for reliability analysis of mechanical components. Stress-Strength Model is defined as the variation in “stress” and “strength” which results in a statistical distribution. If X denotes continuous random variable strength with probability density function $f(x)$ and Y denotes continuous random variable stress with probability density function $f(y)$, X and Y are independent and then the reliability model is given by:

$$R(Y<X)=\int_{-\infty}^{\infty} f(x)\int_{-\infty}^x f(y)dydx \tag{1}$$

$$= \int_{-\infty}^{\infty} f(x)F_y(x)dx \tag{2}$$

The Gamma Distribution

A random quantity X is said to have a gamma distribution with shape α and scale β if its pdf is given by:

$$f(x;\alpha,\beta)=\frac{1}{\Gamma\alpha} \frac{x^{\alpha-1}}{\beta^\alpha} e^{-\frac{x}{\beta}} \quad x>0 \quad \alpha,\beta > 0 \tag{3}$$

The cumulative distribution function is given by:

$$F(x;\alpha,\beta)=\int_0^x \frac{1}{\Gamma\alpha} \frac{x^{\alpha-1}}{\beta^\alpha} e^{-\frac{x}{\beta}} dx \tag{4}$$

Unfortunately, for an arbitrary α , there is no closed form for the distribution function $p = F(x)$ and inverse distribution function $x = F^{-1}(p)$. Further, its

maximum likelihood estimators (MLEs) equations and SSM expression are not evaluated analytically.

Gamma Stress-Strength Model

Let Y (stress) and X (strength) be independent gamma variables whose means are their variance.

The pdf of Y with parameters, a_1 and b_1 , is given by:

$$f(y) = \begin{cases} \frac{b_1^{-a_1} e^{-y/b_1} y^{-1+a_1}}{\text{Gamma}[a_1]} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

The above pdf can be generated through the following Mathematica command:

$$\text{PY} = \text{PDF}[\text{GammaDistribution}[a_1, b_1], y]$$

In addition, the mean and standard deviation can be found using the following commands:

$$\text{Mean}[\text{GammaDistribution}[a_1, b_1]]$$

$$\mu_y = a_1 b_1$$

$$\text{StandardDeviation}[\text{GammaDistribution}[a_1, b_1]]$$

$$\sigma_y = \sqrt{a_1 b_1}$$

The pdf of X with parameters, a_2 and b_2 , is given by:

$$f(x) = \begin{cases} \frac{b_2^{-a_2} e^{-x/b_2} x^{-1+a_2}}{\text{Gamma}[a_2]} & x > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$\mu_x = a_2 b_2$$

$$\sigma_x = \sqrt{a_2 b_2}$$

If a_1 and a_2 are not necessarily integers, Kapur and Lamberson (1977) showed:

$$R = P(Y < X) = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \beta\left(\frac{r}{1+r}, a_1 + a_2\right) \quad (5)$$

Where $r = \frac{b_2}{b_1} \geq 1$, $\Gamma(\cdot)$ is gamma function and $\beta_x(\cdot, \cdot)$ is incomplete beta function. Mathematica, provides single command to evaluate $\Gamma(\cdot)$ by $\text{Gamma}[x]$ and incomplete Beta $[x, a_1, a_2]$. In Table I, II and III, true values of $R = P(Y < X)$ computed from equation 5 with the help of Mathematica are displayed for some selected values of a set of parameters (r, a_1, a_2) .

Table I. The Gamma Stress-Strength Model $R=P(Y<X)$ with $r = 1.0$

$r = 1.0$						
a_2	a_1					
		1	1.5	2.0	2.5	3.0
1.0	0.5	0.353553	0.25	0.176777	0.125	
1.5	0.646447	0.5	0.381282	0.287793	0.215553	
2.0	0.75	0.618718	0.5	0.397748	0.3125	
2.5	0.823223	0.712207	0.602252	0.5	0.408903	
3.0	0.875	0.784447	0.6875	0.591097	0.5	

Table II. The Gamma Stress-Strength Model $R=P(Y<X)$ with $r = 1.5$

$r = 1.5$						
a_2	a_1					
		1.0	1.5	2.0	2.5	3.0
1.0	0.6	0.464758	0.36	0.278855	0.216	
1.5	0.747018	0.62647	0.519334	0.426868	0.348571	
2.0	0.84	0.743613	0.648	0.55771	0.4752	
2.5	0.898807	0.826072	0.747018	0.66639	0.587639	
3.0	0.936	0.88304	0.8208	0.752908	0.68256	

Table III. The Gamma Stress-Strength Model $R=P(Y<X)$ with $r = 2.0$

$r = 2.0$						
a_2	a_1					
		1	1.5	2.0	2.5	3.0
1.0	0.666667	0.544331	0.444444	0.362887	0.296296	
1.5	0.80755	0.708209	0.6151	0.530368	0.454725	
2.0	0.888889	0.816497	0.740741	0.665294	0.592593	
2.5	0.93585	0.886049	0.828933	0.767489	0.704197	
3.0	0.962963	0.929899	0.888889	0.841697	0.790123	

From the tables (I, II, and III), it is evident that if shape parameter (a_2) for strength distribution increases, the reliability of the component also increases.

Maximum Likelihood Estimators (MLEs) of α, β

In the following section, we discussed the estimation of scale and shape parameters through maximum likelihood.

If a random X_1, X_2, \dots, X_n with pdf $f(x; \alpha, \beta)$ is given in equation 3, then the likelihood function is given by:

$$l(\alpha, \beta) = \sum_{i=1}^n [-\log \Gamma(\alpha) - \alpha \log \beta + (\alpha - 1) \log X_i - X_i / \beta] \tag{6}$$

Taking partial derivatives of $\ln(L)$ with respect to α, β , we get:

$$\frac{\partial l}{\partial \alpha} = \sum_{i=1}^n \left[-\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \log \beta + \log X_i \right]$$

$$\frac{\partial l}{\partial \beta} = \sum_{i=1}^n \left[-\frac{\alpha}{\beta} + \frac{X_i}{\beta^2} \right]$$

Setting the second partial derivative equal to zero gives an estimate of scale parameter β as:

$$\hat{\beta}_{MLE} = \frac{\sum_{i=1}^n X_i}{n\hat{\alpha}_{MLE}}$$

A nonlinear equation for the MLE of shape parameter α is obtained if value of estimator β is used.

$$-n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - n \log \frac{\sum_{i=1}^n X_i}{n} + n \log \hat{\alpha}_{MLE} + \sum_{i=1}^n \log X_i = 0$$

Mathematica provides closed solution by following single command (Built-in Wolfram Language Symbol).

$$MLE = \text{FindDistributionParameters}\{\{\text{data}\}, \text{GammaDistribution}[\alpha, \beta]\}$$

Maximum Likelihood Estimator of R

The MLE of Gamma Stress-Strength Model can be obtained by finding MLE from strength random sample (a_2, b_2) and stress random sample (a_1, b_1) respectively.

If two independent random samples, $(X_1, X_2, \dots, X_{n'})$ and $(Y_1, Y_2, \dots, Y_{n'})$, from two gamma distributions are available, then MLE of (a_1, b_1, a_2, b_2) are given by (a_1, b_1, a_2, b_2) . However, from the invariance property of MLE's, the MLE of R is found to be;

$$R = \frac{\Gamma(a_1 + a_2)}{\Gamma(a_1)\Gamma(a_2)} \beta \left(\frac{r}{1+r}, a_1, a_2 \right) \tag{7}$$

Monte Carlo Simulation

In the absence of real data, Monte Carlo Simulation (MCS) is commonly used. It provides artificial data by maintaining the statistical

properties and behavior of real data. The case of gamma distribution simulation is not an easy job due to unavailability of closed form of its inverse function. Following Mathematica command directly provides gamma distributed random samples for a given set of parameters (α, β) and sample size n .

```
RandomVariate[GammaDistribution[ $\alpha, \beta$ ],  $n$ ]
```

Therefore, random strength and stress data can easily be generated for different set of parameters under the assumption of gamma distribution. From these generated data, MLE of a set of parameters (a_1, b_1, a_2, b_2) are obtained by the following command:

```
MLE = FindDistributionParameters[{data}, GammaDistribution[ $\alpha, \beta$ ]]
```

Three different sample sizes ($n = 15, 20, 25$) were used and MLE estimate of R is obtained. The process is repeated 1000 times for each of the sample sizes. In order to see the sample behavior of each of these 1000 estimates of R for various sample sizes, a goodness of fit is performed. Mathematica commands are used to find the goodness of fit. The results for a sample size 20 are shown in Table IV, and it is shown that the R in gamma case follows the beta distribution. Similar results are obtained when simulation is performed for different sets of parameters and sample sizes.

```
d = FindDistribution[data]
DistributionFitTest[data, d, {"TestDataTable", "Kolmogorov-Smirnov"}]
```

Table IV. The best fit results for R

Strength parameters	Stress parameters	R	$E(R)$	n	Simulation
(2.5,3)	(1.5,2)	0.8260	0.8234	20	1000

The Kolmogorov-Smirnov distribution fit test give p -value = 0.473653, where the value of test statistics is 0.0265417.

Conclusion

In conclusion, the Stress-Strength reliability has been estimated in gamma case through Monte Carlo Simulation by implementing Mathematica software. By applying simulation method, it has been found that maximum likelihood estimation of R is unbiased. The sampling distribution of R is Beta distribution, and R lies between 0 and 1. The Beta distribution is a flexible model for the events constrained in an interval (0, 1).

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