NUMERICAL SOLUTION OF PERTURBATION STURM-LIOUVILLE PROBLEMS USING CHEBYSHEV POLYNOMIAL

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Abstract

In this paper, a boundary value problem which consists of the integro-differential equation is considered, Chebyshev polynomial is used to find the numerical solution of perturbation Sturm-Liouville problems, an example of numerical results are given and algorithms are performed by Mathmatica (0.7) program.

Keywords: Sturm-Liouville Problems, Chebyshev polynomial, integro-differential equation, numerical method

Interdiction

The Sturm-Liouville problem is a famous differential equation in pure and applied mathematics. Mathematicians have studied it for over 200 years and highly developed theory and remains an active area of interest. There are a lot of methods for approximating their solutions(Siedlecka,2011; Pryce,1993). Amodio (Amodio,2011) used matrix method for the solution of Sturm-Liouville problems, also Tharwat (Tharwat,2013) find numerical computation of eigenvalues of discontinuous Sturm-Liouville problems with parameter dependent boundary conditions using sinc method, Mehrkanoon (Mehrkanoon,2012) using spline approach to find the solution of Sturm - Liouville problems.

Many authors are studied and solved the Fredholm integro-diferential Equations (Aghazadeh,2009; Khirallah2002). Rabbani (Rabbani,2012) solved Fredholm Integro-Differential Equations System by modified decomposition method. Vahidi (Vahidi,2009) given a numerical solution of Fredholm integro-differential equation by Adomian's decomposition method. Daghman (Daghman,2008) solved an integro-differential equation arising in oscillating magnetic fields using he's homotopy perturbation method and found a numerical solution of fourth order integro differential equations using Chebyshev cardinal functions.

Moreover (Annaby, 2011; Lakcstari, 2010; Al-Mdallal, 2010) studied the spectral of perturbed Sturm-Liouville problem and considered the boundary-value problem which consists of the integro-differential equation.

In this paper we will study the numerical solution of perturbation sturm-Liouville eigenvalue problem of the form:

$$y''(x)+q(x)y(x)+\int_a^b r(t)y(t)dt=\lambda y(x)+f(x), \qquad (1)$$

with the following separate type of conditions, i.e:

$$\begin{cases} \alpha_1 y(a)+\alpha_2 y'(a)=0, \\ \\ \beta_1 y(b)+\beta_2\ y'(b)=0. \end{cases} \qquad (2)$$

Here; $q(x), r(x) \epsilon\ L^1(a,b),\ \lambda \epsilon C\ \ and\ \alpha_i,\ \beta_i\ \epsilon\ R.$ by using the Chebyshev polynomial method.

The Method

In this section we approximate the function *y(x)* in eq(1) by using the first kind Chebyshev polynomial.

Definition 2.1

The Chebyshev polynomials of the first kind can be defined by: (Burden, 2011)

$$T_i(x)=\cos\left(i\ \cos^{-1}x\right) \qquad (3)$$

We let $$y(x)=\sum_{i=0}^{n} a_i T_i(x)\ . \qquad (4)$$

which is equivalent to

$$\begin{cases} T_0(x)=1, \\ T_1(x)=x, \\ T_n(x)=2xT_{n-1}(x)-T_{n-2}(x), \quad n\geq 2. \end{cases}$$

In this method we approximate *y(x)* by substitution the Chebyshev polynomial of the first kind in the form (4) in equation (1) for each $x_i = cos\left(\frac{i\pi}{n}\right)$, $i = 0,1,2,\dots,n$. equation (1) construes to system of linear equation in *n* coefficient. Numerically we will solve the system of linear equation to find the coefficient .

To consider the perturbation Sturm-Liouville eigenvalue problem (1) substituting (4) into (1) we get

$$\sum_{i=0}^{n} a_i\left[T_i''(x)+\sum_{k=0}^{m} q_k C_{k,i}(x)+\sum_{k=0}^{s}{}' r_k D_{k,i}^{a,b}\right]=f(x) \qquad (5)$$

where

$$r_k = \frac{2}{s+1}\sum_{\ell=1}^{s+1} t\left(x_\ell\right) T_k\left(x_\ell\right), \quad k = 0,\ldots,s,$$
$$x_\ell = \cos\left(\frac{2\ell-1}{2(s+1)}\pi\right), \quad \ell = 1,\ldots,s+1. \tag{6}$$

$$T_i''(x) = \frac{(i+1)T_{i-2}(x) - 2iT_i(x) + (i-1)T_{i+2}(x)}{\left(1-x^2\right)^2}, \tag{7}$$

$$C_{k,i}(x) = T_k(x)T_i(x) = \frac{1}{2}\left[T_{k+i}(x) + T_{|k-i|}(x)\right]. \tag{8}$$

and

$$D_{k,i}^{a,b} = \int_a^b C_{k,i}(t)dt = \frac{1}{2}\int_a^b \left[T_{k+i}(t) + T_{|k-i|}(t)\right]dt$$
$$= \frac{1}{2}\left[E_{k+i} + E_{|k-i|}\right] \tag{9}$$

where

$$E_{k+i} = \int_a^b T_{k+i}(t)dt = \left.\begin{cases} \frac{1}{2}\left[\frac{T_{i+k+1}(t)}{i+k+1} - \frac{T_{|i+k-1|}(t)}{i+k-1}\right], & i+k \neq 1 \\ \frac{1}{4}T_2(t) & ,i+k = 1 \end{cases}\right|_a^b, \tag{10}$$

$$E_{|k-i|} = \int_a^b T_{|k-i|}(t)dt = \left.\begin{cases} \frac{1}{2}\left[\frac{T_{|k-i|+1}(t)}{|k-i|+1} - \frac{T_{||k-i|-1|}(t)}{|k-i|-1}\right], & |k-i| \neq 1 \\ \frac{1}{4}T_2(t) & ,|k-i| = 1 \end{cases}\right|_a^b. \tag{11}$$

Substituting (4) in (2) yields

$$\left.\begin{aligned} \alpha_1\sum_{i=0}^{n} a_i T_i(a) + \alpha_2\sum_{i=0}^{n} a_i T_i'(a) = 0 \\ \beta_1\sum_{i=0}^{n} a_i T_i(b) + \beta_2\sum_{i=0}^{n} a_i T_i'(b) = 0 \end{aligned}\right\} \tag{12}$$

Using $x_i \in(-1,1)$, $i=1,\ldots,n-2$ as the collocation points into (5) and (11) we obtain the following system:

$$\left.\begin{aligned} &\sum_{i=0}^{n} a_i \left[T_i''(x_k) + \sum_{k=0}^{m} q_k C_{k,i}(x_k) + \sum_{k=0}^{s} r_k D_{k,i}^{a,b} \right] = f(x_k) \\ &\qquad (k = 1,2,\dots,n-2) \\ &\alpha_1 \sum_{i=0}^{n} a_i T_i(a) + \alpha_2 \sum_{i=0}^{n} a_i T_i'(a) = 0 \\ &\beta_1 \sum_{i=0}^{n} a_i T_i(b) + \beta_2 \sum_{i=0}^{n} a_i T_i'(b) = 0 \end{aligned}\right\} \qquad (13)$$

for each $x_i = cos\left(\frac{i\pi}{n}\right)$, $i = 0,1,2,\dots,n$ we get system of linear equations (13) in n coefficient. Numerically we will solve the system of linear equations to find the coefficient.

The system (13) does not always give a unique solution for the coefficient c_i 's. In order to maintain uniqueness for the solution of this problem, the boundary conditions in eq. (2) are used.

Here, the algorithm (13) is performed by Mathmatica (0.7) program.

Transforming the Interval

It is sometimes necessary to take a problem studied on an interval [a,b], then we convert the variable so that the problem is reformulated on [-1,1]. The change of variable

$$x = \left(\frac{b-a}{2}\right) z + \frac{b+a}{2}. \qquad (14)$$

converts the interval $-1 \le z \le 1$ to $a \le x \le b$, conversely

$$z = 2\left(\frac{x-a}{b-a}\right) - 1 \qquad (15)$$

Transform the points $a \le x \le b$ to $-1 \le z \le 1$.

Numerical result

Consider the following equation

$$y''(x) + x\, y(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} (2x+1) y(x)\, dx = x^4 + 7x + \frac{41}{40}, \qquad (16)$$

with the conditions

$$\left.\begin{aligned} &-\frac{1}{7} y\left(-\frac{1}{2}\right) + \frac{1}{6} y'\left(-\frac{1}{2}\right) = 0 \\ &-\frac{1}{9} y\left(\frac{1}{2}\right) + \frac{1}{6} y'\left(\frac{1}{2}\right) = 0 \end{aligned}\right\} \qquad (17)$$

To solve this problem we use algorithm (13) and perform it by Mathmatica (0.7) program to get the result

x_i	Exact	Numerical
-0.5	0.875	0.875
-0.4	0.936	0.936
-0.3	0.973	0.973
-0.2	0.992	0.992
-0.1	0.999	0.999
0	1	1
0.1	1.001	1.001
0.2	1.008	1.008
0.3	1.027	1.027
0.4	1.064	1.064
0.5	1.125	1.125

Table (1)

If we let x_1= -0.5, x_2=0 and x_3=0.5 in the system (13) and use the boundary conditions we get the following system

$$\left.\begin{array}{l}
\frac{1}{2}a_0+\frac{5}{12}a_1+\frac{41}{12}a_2-\frac{129}{10}a_3+\frac{521}{60}a_4=-\frac{193}{80},\\
a_0+\frac{1}{6}a_1+\frac{19}{6}a_2-\frac{2}{5}a_3-\frac{467}{30}a_4=\frac{41}{40},\\
\frac{3}{2}a_0+\frac{5}{12}a_1+\frac{35}{12}a_2+\frac{111}{10}a_3+\frac{491}{60}a_4=\frac{367}{80},\\
-\frac{1}{7}a_0+\frac{5}{21}a_1-\frac{11}{42}a_2-\frac{1}{7}a_3-\frac{53}{42}a_4=0,\\
-\frac{1}{9}a_0+\frac{1}{9}a_1+\frac{13}{18}a_2+\frac{1}{9}a_3-\frac{1}{18}a_4=0.
\end{array}\right\} \qquad (18)$$

The solution of the above system by using Mathmatica (0.7) program is

$$\left.a_0=1,\ a_1=\frac{3}{4},\ a_2=0,\ a_3=\frac{1}{4},\ a_4=0\right\}. \qquad \mathbf{(19)}$$

Substituting (19) in (4) we get the numerical solution of equation (16)

$$y(x)=1+x^3$$

which is identical to the exact solution.

Conclusions

The boundary value problem which consists of the integro-differential equation - Sturm-Liouville Problems- is solved numerically by using Chebyshev polynomial. Examples show that our method is very effective and efficient. Moreover, our proposed method provides highly accurate results.

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