GLOBAL SOLUTIONS OF THE FUCHSIAN-CAUCHY PROBLEM IN GEVREY SPACES

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Abstract:
We consider the Fuchsian Cauchy problem associated to linear partial differential equations with Fuchsian principal part of order m and weight µ in the sense of M. S. Baouendi and C. Goulaouic [2]. We obtain existence and uniqueness of a global solution to this problem in the space of holomorphic functions with respect to the fuchsian variable t and in Gevrey spaces with respect to the other variable x. The method of proof is based on the application of the fixed point theorem in some Banach spaces defined by majorant functions that are suitables to this kind of equations. We introduce new majorant functions as in [4]and [5] which allow us to simplify the proof given in [3].

Key Words: Fuchsian linear partial differential equations, Global solution, Gevrey spaces, Method of majorants, Fixed point theorem

Introduction

In [1], is established by J.M.Bony and P.Schapira that the hyperbolicity of the operator is sufficient so that the Cauchy problem is well posed in the class of holomorphic functions (Bony-Schapira-Theorem). But, this result was established by C.Wagschal in [10] for operators not necessarily hyperbolics.

To demonstrate his local existence theorem for an holomorphic solution of a semilinear operator, A.Cauchy used his method known as the "Majorant series method" which consists to find the solution of the problem in the form of a serie and to prove the convergence of this serie by bounding above the modulus of the coefficients of this serie by those of another serie whose terms are positive and is convergent.

C. Wagschal [10] simplified local resolution of the nonlinear Goursat problem in spaces of holomorphic functions and in Gevrey spaces to the fixed point theorem. His technique consists to define Banach algebras, either through the formalism of majorant functions of A.Cauchy in the holomorphic case, either through the formalism of formal series in the Gevrey case, in which his problem is reduced to the search for fixed points of some map that he builts from the original problem and he shows strictly contracting in balls of these algebras.

And since the method of fixed point developed by C.Wagschal [10] has been used successfully in other problems. It cites the work of P.Pongérard and C.Wagschal [7] its aim is to reduce the global resolution of the Cauchy-Kowalvski problem to the fixed point theorem in spaces of entire functions and entire functions of finite order.

Interest was directed later to study or to simplify the study of Fuchsian equations by the method of fixed point. The major difficulty is in the search for Banach spaces, so for majorant functions, suitable for the study of this type of equations.

P.Pongérard [9] extended the conclusions of [7] to Fuchsian operators in the class of entire functions and entire functions of finite order using this method of fixed point.

In [4], we have simplified the proof given by P.Pongérard in [9] by defining new majorant functions by introducing a new parameter ρ and we generalized his result to a differential operator with several Fuchsian variables.

In [3], the authors have extended the result of [9] to a global resolution in spaces of holomorphic functions with respect to the Fuchsian variable and in Gevrey classes with respect to the other variables. One of these classes of Gevrey is more general than the class introduced in [10] and the second is introduced by H. Komatsu in [6].

In this work, we simplify the proof given in [3] using the same technique introduced in [4] by defining new majorant functions by introducing a new parameter ρ. Thus, we establish a global solution for our problem which is holomorphic with respect to the Fuchsian variable and is in Gevrey spaces with respect to the other variable. Our study is limited to Gevrey class defined by H.Komatsu [6]. This same technique has enabled us in [5] to give global resolution for some nonlinear equations of Fuchs type in these same Gevrey classes.

The Problem Formulation And Result

We study Fuchsian linear partial differential equations in the space $\mathbb{C}\times\mathbb{R}^n$. We denote by t the generic point of $\mathbb{C}$ and by $x=(x_1,\dots,x_n)$ the generic point of $\mathbb{R}^n$. Let $'\Omega$ be an open set in $\mathbb{R}^n$. For a multiindice $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$ we denote $D^\alpha=D_1^{\alpha_1}\dots D_n^{\alpha_n}$ where $D_j=D_{x_j}$ is the partial derivative with respect to x_j and by $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$.

Let $m\geq 1$ be an integer, we denote by E a subset of $\{(l,\alpha)\epsilon\mathbb{N}\times\mathbb{N}^n;\ l+|\alpha|\leq m,\ l<m,\ \alpha\neq 0\}$.

Let $0\leq\mu\leq m$, we consider the Cauchy problem

$$\begin{cases} a(t,D_t)u(t,x)=\displaystyle\sum_{(l,\alpha)\epsilon E} a_{(l,\alpha)}(t,x)\,t^{\upsilon+1+l-\mu}\,D_t^l D^\alpha u(t,x)\ +f(t,x); & (t,x)\in\mathbb{C}\times{}'\Omega, \\ D_t^j u(t,x)=w_j(x), \qquad 0\leq j<\mu, \qquad x\in{}'\Omega, \end{cases} \qquad (1)$$

where $a(t,D_t)$ is the linear differential operator defined by $a(t,D_t)=\sum_{l=\mu}^m a_l t^{l-\mu} D_t^l$, and a_l for $\mu\leq l\leq m$ are complex constants with $a_m\neq 0$.

$a(t,D_t)$ is then a Fuchsian principal part of order m and weight μ.

$\upsilon=\upsilon(l)$ is the integer number defined by $\upsilon=\max(\mu-l-1,0)$ and the coefficients $a_{(l,\alpha)}$ for $(l,\alpha)\epsilon E$ are polynomial functions with respect to x of order strictly inferior to $|\alpha|$ with holomorphic coefficients in $\mathbb{C}_t$. It means that

$$for\ (l,\alpha)\epsilon E,\ \ a_{(l,\alpha)}(t,x) = \sum_{|\beta|<|\alpha|} a_{l\alpha\beta}(t)\,x^\beta\ \ where\ a_{l\alpha\beta}\ is\ an\ holomorphic\ function\ in\ \mathbb{C}_t. \qquad (2)$$

We associate to the operator $a(t,D_t)$ the polynomial $P(\lambda)=\sum_{l=\mu}^m a_l\prod_{j=0}^{l-1}(\lambda-j)$ and we consider $\prod_\phi=1$.

We obtain: $t^\mu\,a(t,D_t)=\sum_{l=\mu}^m a_l t^l D_t^l=P(tD_t)$. Then $P(tD_t)$ is a Fuchsian principal part of weight 0.

For the choice of f and w_j $(0\leq j<\mu)$, we introduce the following definitions. We recall the Gevrey class definition in the sense given by H.Komatsu in [6]

Definition 1: Let $d\geq 1$. A function $v\in C^\infty('\Omega)$ is said to be in the Gevrey class $G^{(d)}$ if for every compact set $K\subset{}'\Omega$ and $h>0$ there exists a constant $c=c_{K,h}\geq 0$ such that

$$\forall\alpha\in\mathbb{N}^n,\ \ sup_{x\in K}|D^\alpha v(x)|\ \leq\ c\,h^{|\alpha|}\,(|\alpha|!)^d.$$

Examples:

1. For $n\in\mathbb{N}$; $f_n(x)=sin\,nx\ \in G^{(d)}(\mathbb{R})$. Then the vector space generated by $\{sin\,nx, cos\,nx,\ n\in\mathbb{N}\}$ is a subset of $G^{(d)}(\mathbb{R})$.
2. For $b\in\mathbb{N}$; $x\in\mathbb{R}^n$, $L(x)=x^\beta$ a polynomial of order $\beta\in\mathbb{N}^n$ such that $|\beta|\leq b$, we obtain $L(x)\in G^{(d)}(\mathbb{R}^n)$. Then the set $\mathbb{R}[x]$ of polynomials is a subet of $G^{(d)}(\mathbb{R}^n)$.

For U an open set in $\mathbb{C}$, we denote by $C^{\omega,\infty}(U\times{}'\Omega)$ the algebra of functions $u: U\times{}'\Omega\to\mathbb{C}$ which admit derivatives for every order with respect to x, are continuous in $U\times{}'\Omega$ and holomorphic with respect to t.

Definition 2: We say that a function u is of $G^{(\omega,d)}(U\times{}'\Omega)$ class if u belongs to $C^{\omega,\infty}(U\times{}'\Omega)$ and if for every compact set $K\subset{}'\Omega$ and $h>0$ there exists a constant $c=c_{K,h}\geq 0$ such that

$$\forall\alpha\in\mathbb{N}^n,\qquad \forall t\in U,\qquad \forall x\in K;\qquad |D^\alpha u(t,x)|\ \leq\ c\,h^{|\alpha|}\,(|\alpha|!)^d.$$

For $R > 0$, we denote by $D_R = \{t \in \mathbb{C};\ \ |t| < R\}$.

$G^{(\omega,d)}(\mathbb{C} \times {}'\Omega)$ denotes the set of functions $u \in C^{\omega,\infty}(\mathbb{C} \times {}'\Omega)$ such that for all $R > 0$, $u \in G^{(\omega,d)}(D_R \times {}'\Omega)$.

$G^{(\omega,d)}(\mathbb{C} \times {}'\Omega)$ is a sub-algebra of $C^{\omega,\infty}(U \times {}'\Omega)$.

The coefficients $a_{(l,\alpha)}$ assumed verifying (2), then we obtain the following theorem:

Theorem 1: If $P(\lambda) \neq 0$ for every integer $\lambda \geq \mu$, then for any functions $w_j \in G^{(d)}({}'\Omega)$, $(0 \leq j \leq \mu)$ and $f \in G^{(\omega,d)}(\mathbb{C} \times {}'\Omega)$; the Cauchy problem (1) admits a unique solution $u \in G^{(\omega,d)}(\mathbb{C} \times {}'\Omega)$.

Remark 1:

1. The theorem1 establishes that the solution of the problem (1) inherits the $G^{(d)}$ Gevrey regularity of the data w_j, $(0 \leq j \leq \mu)$.
1) If ${}'\Omega$ is a bounded open set in $\mathbb{R}^n$, then we can extend the same study for Fuchsian operators where the coefficients $a_{(l,\alpha)}$ are polynomials of any order $N_{l,\alpha}$. In this case, the hypothesis (2) is written in the form:

$$for\ (l,\alpha) \epsilon E,\ \ a_{(l,\alpha)}(t,x) = \sum_{|\beta| < N_{l,\alpha}} a_{l\alpha\beta}(t)\, x^{\beta}\ \ where\ a_{l\alpha\beta}\ is\ an\ holomorphic\ function\ in\ \mathbb{C}_t.$$

Proof of theorem 1

A. ***Reduction of the Cauchy problem (1)***

We set

$$P_1(u) = a(t,D_t)u - \textstyle\sum_{(l,\alpha)\epsilon E} a_{(l,\alpha)}(t,x)\, t^{\nu+1+l-\mu}\, D_t^l D^{\alpha} u.$$

Then by denoting $v(t,x) = u(t,x) - \sum_{j=0}^{\mu-1} \frac{t^j}{j!} w_j(x)$ and $g = f - P_1(\sum_{j=0}^{\mu-1} \frac{t^j}{j!} w_j)$; the Cauchy problem (1) is equivalent to the problem $\begin{cases} P_1(v) = g \\ D_t^j v(t,x) = w_j(x),\ \ 0 \leq j < \mu. \end{cases}$

By a second change of unknown: $v(t,x) = t^{\mu} z(t,x)$, the previous problem is reduced to solving the equation: $P_1(t^{\mu} z) = g$.

Using the relation: $t^{\mu} a(t,D_t)\,(t^{\mu} z) = P(tD_t)(t^{\mu} z) = t^{\mu}\, P(tD_t + \mu)z$; the Cauchy problem (1) is reduced to the resolution of the equation

$$P(tD_t)z = \sum_{(l,\alpha)\epsilon E} a_{(l,\alpha)}(t,x)\, t^{\nu+1+l-\mu}\, D_t^l t^{\mu} D^{\alpha} z + g \qquad (3)$$

and its solution is given by: $u = \sum_{j=0}^{\mu-1} \frac{t^j}{j!} w_j + t^{\mu} z$.

The coefficients $a_{(l,\alpha)}$ always verify the hypothesis (2). $P(tD_t)$ is a Fuchsian principal part of weight 0 which verifies $P(\lambda) \neq 0$ for every $\lambda \in \mathbb{N}$.

For proving that $g \in G^{(\omega,d)}(\mathbb{C} \times {}'\Omega)$ we need the following lemma.

Lemma 1: Let $R > 0$ and $v \in G^{(\omega,d)}(D_R \times {}'\Omega)$. Then for every $\alpha \in \mathbb{N}^n$, $D^{\alpha} v \in G^{(\omega,d)}(D_R \times {}'\Omega)$.

For the reduction of the equation (3) we use the following lemma

Lemma 2:

1. $\exists c_0 > 0$ such that $P(\lambda) \geq c_0\, max(1, \lambda^m)$ for every $\lambda \in \mathbb{N}$.
2. For any $R > 0$, the operator $P(tD_t)$ is an automophism of the vector space $G^{(\omega,d)}(D_{R'} \times {}'\Omega)$ for every $0 < R' < R$. Its inverse is defined by:

$$P^{-1}(u)(t,x) = \sum_{k\in\mathbb{N}} \frac{t^k}{k!} \frac{D_t^k u(0,x)}{P(k)}.$$

In replacing z by $P^{-1}(z)$ in equation (3) and noting again z by u, then equation (3) is equivalent to:

$$u = \sum_{(l,\alpha)\epsilon E} a_{(l,\alpha)}(t,x)\, t^{\upsilon+1+l-\mu}\, D_t^l t^\mu D^\alpha P^{-1}(u) + g \tag{4}$$

Thus, the resolution of the Cauchy problem (1) is reduced to looking for the fixed points of the map F defined by:

$$F(u) = H(u) + g \tag{5}$$

where the operator H is defined by

$$H(u) = \sum_{(l,\alpha)\epsilon E} a_{(l,\alpha)}(t,x)\, t^{\upsilon+1+l-\mu}\, D_t^l t^\mu D^\alpha P^{-1}(u) \tag{6}$$

This takes us to introduce a Banach space associated to a majorant function where we establish that the map F is strictly contracting so we can apply the Banach fixed point theorem. We bring a new majorant function in this work compared to [3], we consider the majorant function of [9] and we apply the same technique introduced in [4], it concerns the introduction of the parameter ρ which allows us to simplify the proof presented in [3].

B. ***Banach space*** $C^{\omega,\infty}_{\Phi^d_{\rho,\rho R,\zeta}}(\Omega_R)$

Let D be an open neighborhood of the origine in $\mathbb{C}$ and Φ be the formal serie $\Phi = \Phi(t,x) = \sum_{\alpha\in\mathbb{N}^n} \Phi_\alpha(t)\frac{x^\alpha}{\alpha!}$ where Φ_α is an entire serie $\gg 0$ which converges in D.

For any function $u \in C^{\omega,\infty}(U \times {}'\Omega)$, the relation $u \ll \Phi$ is defined in [10] by

$$u \ll \Phi \Leftrightarrow (\forall\alpha \in \mathbb{N}^n), \quad (\forall x \in \Omega), \quad D^\alpha u(t,x) \ll \Phi_\alpha(t).$$

Let $R > 0$. For a parameter $\rho > 0$, we denote by $D_{\rho R} = \{t \in \mathbb{C};\ \rho|t| < R\}$, $\Omega_{\rho,R} = D_{\rho,R} \times \Omega$ and we denote by convention $D_{\rho\rho,R} = D_R$; $\Omega_{\rho,\rho R} = \Omega_R$.

For a parameter $\zeta = (\zeta_1, \dots, \zeta_n) \in (\mathbb{R}_+^*)^n$ and a given integer $s \geq m$, we consider the Gevrey formal serie

$$\Phi^d_{\rho,\rho R,\zeta} = \Phi^d_{\rho,\rho R,\zeta}(t,x) = \sum_{p\in\mathbb{N}} (\rho t)^p\, (\rho R)^{s'p}\, \frac{\left(D^{sp}\phi_{\rho R,\zeta}\right)^d(\zeta.x)}{(sp)!} \tag{7}$$

where $\zeta.x = (\zeta_1.x_1, \dots, \zeta_n.x_n)$; $s' = s - 1 \geq 0$; ϕ is the majorant function defined by

$$\phi_{\rho R,\zeta}(\zeta.x) = e^{\rho^{-1}(\zeta.x)}\frac{1}{\rho R - \zeta.x}. \tag{8}$$

We denote by $C^{\omega,\infty}_{\Phi^d_{\rho,\rho R,\zeta}}(D_R \times {}'\Omega)$ the space of functions $u \in C^{\omega,\infty}(D_R \times {}'\Omega)$ such that $\exists c \geq 0;\ u \ll \Phi^d_{\rho,\rho R,\zeta}$.

$C^{\omega,\infty}_{\Phi^d_{\rho,\rho R,\zeta}}(D_R \times {}'\Omega)$ with the norm $||u||_{\Phi^d} = \min\{c \geq 0; \quad u \ll c\,\Phi^d_{\rho,\rho R,\zeta}\}$ is a Banach space.

C. ***Proof of the contracting of*** F

For every compact set K in $'\Omega$ of non-empty interior K°, we denote $K_R = D_R \times K^\circ$.

If $\zeta = (\zeta_1, \dots, \zeta_n)$; $\zeta' = (\zeta'_1, \dots, \zeta'_n) \in (\mathbb{R}_+^*)^n$, we write $\zeta \leq \zeta'$ if $\zeta_j \leq \zeta'_j$ for every $1 \leq j \leq n$ and we write $\zeta < \zeta'$ if $(\zeta \leq \zeta'\ and\ \zeta \neq \zeta')$.

Assume that the coefficients $a_{(l,\alpha)}$ verify the hypothesis (2) then for any function $g \in G^{(\omega,d)}(\mathbb{C} \times {}'\Omega)$ and using some intermediate results not stated in this paper , we prove the following proposition.

Proposition 1: Let K be a fixed compact set in $'\Omega$ of non-empty interior K° and let $R > 0$ be fixed. Then, for $\rho > 0$ and $\zeta \in (\mathbb{R}_+^*)^n$ verifying $\rho R > 1$ and $\zeta = (\zeta_0, \dots, \zeta_0) < (1, \dots, 1)$, there exists $\rho_0 > 0$ and a constant $c \in]0,1[$ such that: for any $\rho \geq \rho_0$, there exists $b_\rho > 0$ for which

$$\forall b \geq b_\rho; \quad F\big(B(0,b)\big) \subset B(0,b) \subset C^{\omega,\infty}_{\Phi^d_{\rho,\rho R,\zeta}}(K_R) \tag{9}$$

$$\forall u, u' \in C^{\omega,\infty}_{\Phi^d_{\rho,\rho R,\zeta}}(K_R); \quad \left||F(u) - F(u')|\right|_{\Phi^d} \leq c \left||u - u'|\right|_{\Phi^d} \qquad (10)$$

where $B(0, b)$ is the closed ball of center 0 and radius b of $C^{\omega,\infty}_{\Phi^d_{\rho,\rho R,\zeta}}(K_R)$.

Under the hypotheses of this proposition, we deduce from the fixed point theorem that the map F admits a unique fixed point in $C^{\omega,\infty}_{\Phi^d_{\rho,\rho R,\zeta}}(K_R)$.

D. ***Construction of the fixed point of F in $'\Omega_R$***

In the rest of the proof we fix $\zeta = (\zeta_0, \dots, \zeta_0) < (1, \dots, 1)$.

For the construction of the fixed point of F in $'\Omega_R$, we use the three following statements.

Lemma 3: Let $R > 0$ and let K be a compact set in $'\Omega$ of non-empty interior K°. For any $\rho, \rho' > 0$ such that $\rho \geq \rho'$ we have

$$C^{\omega,\infty}_{\Phi^d_{\rho,\rho R,\zeta}}(K_R) \subset C^{\omega,\infty}_{\Phi^d_{\rho',\rho' R,\zeta}}(K_R)$$

and the canonical function of the inclusion is continuous of norm inferior to 1.

Proposition 2: Let $(K_j)_{j\in\mathbb{N}}$ be an exhaustive sequence of compact sets in $'\Omega$, then there exist an increasing sequence of positive numbers $\rho_j = \rho_{K_j}$ and a sequence $(u_j)_{j\in\mathbb{N}}$ of fixed points of F such that

i) u_j is unique in $C^{\omega,\infty}_{\Phi^d_{\rho_j,\rho_j R,\zeta}}(D_R \times K_j^\circ)$,

ii) $u_j \in \bigcap_{\rho\geq\rho_j} C^{\omega,\infty}_{\Phi^d_{\rho,\rho R,\zeta}}(D_R \times K_j^\circ)$.

Lemma 4: There exists a unique fixed point u of the strictly contracting F defined in $D_R \times \Omega$ satisfying

$$(\forall j \in \mathbb{N}), \qquad u \,/\, K_j^\circ \in \bigcap_{\rho\geq\rho_j} C^{\omega,\infty}_{\Phi^d_{\rho,\rho R,\zeta}}(D_R \times K_j^\circ).$$

Next, we prove that this solution u defined in $D_R \times \Omega$ in lemma 4 is in the class $G^{(\omega,d)}(D_R \times {}'\Omega)$. Then, using the priciple of analytic continuation we show for every $R > 0$, the uniqueness of this fixed point of F in the class $G^{(\omega,d)}(D_R \times {}'\Omega)$.

E. ***End of the proof of theorem 1***

For every $R > 0$, the map F admits a unique fixed point u_R in $G^{(\omega,d)}(D_R \times {}'\Omega)$. Then, using the reattachment of the solutions u_R, we define a unique fixed point of F in $G^{(\omega,d)}(\mathbb{C} \times {}'\Omega)$.

To complete the proof of the theorem 1, we take $t^\mu u + \sum_{j=0}^{\mu-1} \frac{t^j}{j!} w_j$ as the unique solution in $G^{(\omega,d)}(\mathbb{C} \times {}'\Omega)$ of the Cauchy problem (1).

Conclusion

This technique of introducing the parameter $\rho > 0$, allows us to define new majorant functions in this paper to simplify the proof given in [3]. We have also used this same technique in [5] with success to study some nonlinear equations of Fuchs type in this same class of Gevrey. Also we have used this technique in [4] to simplify the proof of [9] for linear Fuchsian operators with several variables.

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