ALGEBRAIC SCHUR COMPLEMENT APPROACH FOR A NON LINEAR 2D ADVECTION DIFFUSION EQUATION

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Abstract:

This work deals with a domain decomposition approach for non stationary non linear advection diffusion equation. The domain of calculation is decomposed into q≥2 non-overlapping sub-domains. On each sub-domain the linear part of the equation is descretized using implicit finite volumes scheme and the non linear advection term is integrated explicitly into the scheme. As non-overlapping domain decomposition, we propose the Schur Complement (SC) Method. The proposed approach is applied for solving the local boundary sub-problems. The numerical experiments applied to Burgers equation show the interest of the method compared to the global calculation. The proposed algorithm has both the properties of stability and efficiency. It can be applied to more general non linear PDEs and can be adapted to different FV schemes.

Key Words: Non linear advection-diffusion problems, Structured mesh, Burgers equation, Finite volumes method (FVM), Schur Complement (SC)

The system of equations

Let us consider the following initial boundary value problem:

Find c: $\Omega \times (0,T) \to \mathbb{R}$ such that

$$(1.1)\qquad \begin{cases} \dfrac{\partial c}{\partial t} - \nu\,\Delta c + \displaystyle\sum_{s=1}^{2} \dfrac{\partial f_s(c)}{\partial x_s} = g & in\ \ \Omega\times(0,T) \\ c(x,t) = c_D(x,t) & on\ \ \partial\Omega\times(0,T) \\ c(x,0) = c_0(x) & in\ \Omega \end{cases}$$

Where $\Omega \subseteq \mathbb{R}^2$ is a bounded polygonal domain and (0; T), where T > 0, time interval. By $\bar{\Omega}$ and $\partial\Omega$ we denote the closure and boundary of Ω, respectively.

We assume that the data have the following properties [6, 7, 8]:

a) $f_s \in C^1(\mathbb{R}),\ f_s(0) = 0,\ |f_s| \le C_{f'},\ s = 1,2,$

b) $\nu > 0,$

c) $g \in C([0,T];L^2(\Omega))$

d) C_D is the trace of some $C^* \in C([0,T];H^1(\Omega)) \cap L^\infty(\Omega\times(0,T))$ $on\ \partial\Omega\times(0,T),$

e) $c_0 \in L^2(\Omega)$.

In virtue of assumption a), the functions f_s satisfy the Lipschitz condition with constant $C_{f'}$, the functions f_s are fluxes of the quantity c in the direction x_s, its represent convective terms, the constant $\nu > 0$ is the diffusion coefficient.

We use the standard notation for function spaces (see, e.g. [9]): $L^p(\Omega)$, $L^p(\Omega\times(0,T))$ denote the Lebesgue spaces, $W^{k,p}(\Omega)$, $H^k(\Omega) = W^{k,2}(\Omega)$ are the Sobolev spaces, $L^p(0,T;X)$ is the Bochner space of functions p-integrable over the interval (0, *T*) with values in a Banach space *X*,

$C([0,T];X)$ $(C^1([0,T];X))$ is the space of continuous (continuously differentiable) mappings of the interval $[0,T]$ into X.

We shall assume that problem (1.1) has a weak solution (cf. [6,7]), satisfying the regularity conditions:

$$(1.2) \qquad c,\ \frac{\partial c}{\partial t},\ \frac{\partial^2 c}{\partial t^2} \in L^\infty(0,T;H^{p+1}(\Omega))$$

where an integer p ≥ 1 will denote a given degree of polynomial approximations. Such a solution satisfies problem (1.1) pointwise. Under (1.2),

$$c \in C([0,T];H^{p+1}(\Omega)), \ \ and \ \ \frac{\partial c}{\partial t} \in C([0,T];L^2(\Omega))$$

Finite volume approach

The finite volumes approach consists in dividing the domain of calculation Ω into a finite number of control volumes (CVs) V_i (i=1,.., N×M) with $\Omega = \bigcup_{i=1}^{N\times M} V_i$.

For a general CV we use the notation of the distinguished points (mid-point, midpoints of faces) and the unit normal vectors according to the notation as indicated in Figure 1 (right) . The midpoints of neighboring CVs we denote with capital letters W, S, etc. (see Figure 1 left), these notations are given in [3].

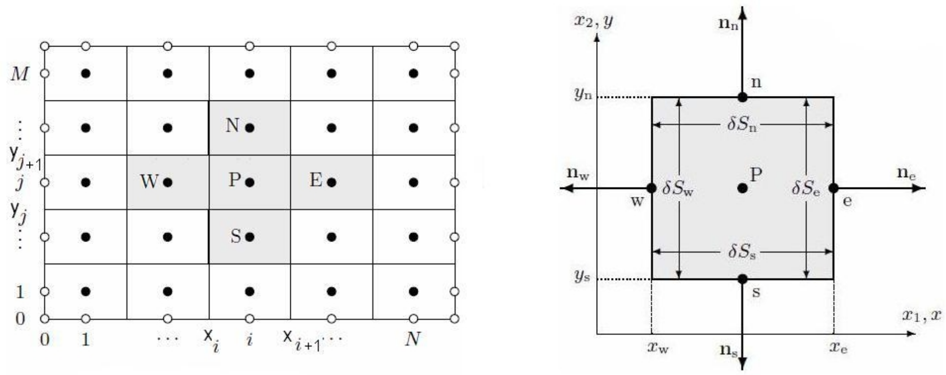

Figure 1. FV structured mesh of domain Ω

By integrating the equation (1.1) over an arbitrary CV V_P and applying the Green formula, we obtain:

$$(2.1) \qquad \int_{V_P} \frac{\partial c}{\partial t}(t)dV_P + \sum_a \int_{S_{a,P}} \sum_{s=1}^{2} f_s(c(t))n_{a,s}dS_{a,P} - \nu \sum_a \int_{S_{a,P}} \nabla c(t) n_a dS_{a,P} = \int_{V_P} g(t)dV_P,$$

where $S_{a,P}$ $(a = e,n,w,s)$ are the four faces of volume V_P (see Figure 1), $n_a = (n_{a,1}, n_{a,2})$ are the unit normal vectors to the face $S_{a,P}$ and $\mu(V_P)$ is the volume of cell V_P.

Approximating the linear operator $\partial_t - \nu\Delta$ by the implicit Euler method and the non-linear term by an explicit approximation, we get:

$$(2.2) \qquad \mu(V_P)\frac{c_P^{n+1} - c_P^n}{\Delta t} + \sum_a \int_{S_{a,P}} \sum_{s=1}^{2} f_s(c^n)n_{a,s}dS_{a,P} - \nu \sum_a \int_{S_{a,P}} \nabla c^{n+1} n_a dS_{a,P} = \mu(V_P)g_P^n,$$

where

$$g_P^n = \frac{1}{\mu(V_P)} \int_{V_P} g(x,t^n)dV_P,$$

and

$$c_p^0 = \frac{1}{\mu(V_P)} \int_{V_P} c_0(x)dV_P, \quad or \quad c_P^0 = c_0(x_P).$$

- For the discretization of diffusion term, we have considered a centred difference scheme.
- For the convective terms we use the numerical flux, for the CV V_P and $S_{a,P}$ $(a = e, n, w, s)$:

$$(2.3) \qquad \sum_{s=1}^{2} f_s(c^n) n_{a,s} = \begin{cases} \sum_{s=1}^{2} f_s(c_P^n) n_{a,s} & \textit{if} \quad K > 0, \\ \sum_{s=1}^{2} f_s(c_I^n) n_{a,s} \quad (I = E, W, S, N) & \textit{if} \quad K \le 0, \end{cases}$$

where

$$K = \sum_{s=1}^{2} f_s(\overline{c}^n) n_{a,s}, \quad \overline{c}^n = \frac{1}{2}(c_P^n + c_I^n).$$

- For the approximation of the volume and surface integrals, we have employed the midpoint rule.

Let us denote that c_I^n is the concentration on the volume V_P (*I*=*P*, *E*, *W*, *N* or *S*) at time t_n. The concentration variables c_I^{n+1} and c_I^n (*I*=*P*, *E*, *W*, *N* or *S*) in equation (2.2) can be arranged as follows:

$$(2.4) \qquad a_P c_P^{n+1} + a_E c_E^{n+1} + a_W c_W^{n+1} + a_N c_N^{n+1} + a_S c_S^{n+1} = b_P \, ,$$

b_P is a constant depending on, the source term g_P^n, c_P^n, the discretized convection flux, the boundary and the initial conditions.

Finally, the numerical scheme is expressed as the linear system:

$$AC_P^{n+1} = b,$$

where A is a (N × M , N × M) type matrix of coefficients a_I (*I*=*P*, *E*, *W*, *N* or *S*), C_P^{n+1} and *b* are the vectors of c_P^{n+1} and b_P respectively.

Schur complement method

Domain decomposition

The domain Ω is decomposed into multi-domain nonoverlapping strip decomposition $\Omega_1, \dots, \Omega_q$ where $\overline{\Omega} = \cup_{i=1}^{q} \overline{\Omega}_i$ and $\Omega_i \cap \Omega_j = \emptyset$ when $i \neq j$ (figure 2).

Let Γ_{ij} denote the interface between Ω_i and Ω_j and $\Gamma = \cup \, \Gamma_{ij}$, and by n^i the normal direction (oriented outward) on Γ_{ij} for i=1, …, q-1 and j=i+1.

For simplicity of notation we also set $n = n^i$.

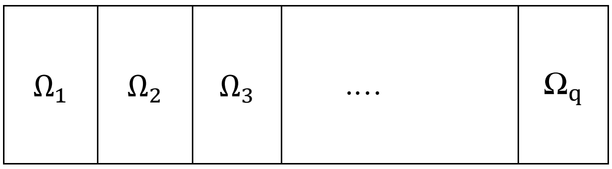

Figure 2. Non-overlapping strip decomposition

Considering a rectangular mesh of Ω, each subdomain Ω_i is partitioned into n_i (i=1,…,q) cells in X direction and m cells in Y direction (figure 3).

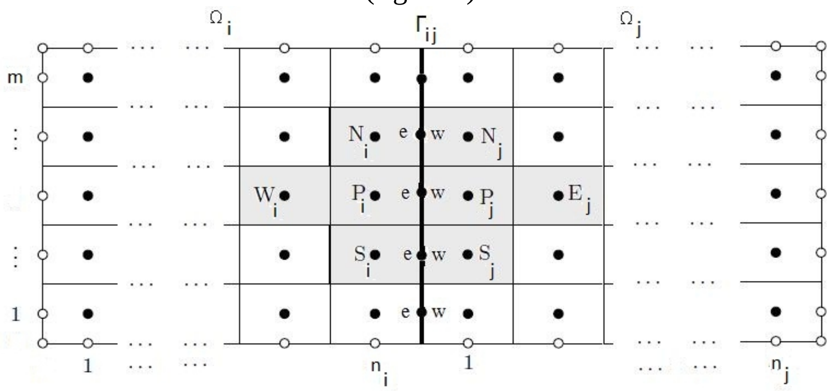

Figure 3. Domain decomposition and structured conforming mesh of domain Ω

The problem (1.1) can then be expressed as :

$$
(3.1)\quad \begin{cases}
\dfrac{\partial c_i}{\partial t} - \nu\,\Delta c_i + \displaystyle\sum_{s=1}^{2} \dfrac{\partial f_s(c_i)}{\partial x_s} = g & in\ \Omega_i \times (0,T), i = 1,...,q \\
c_i(x,t) = c_D(x,t) & on\ \ (\partial\Omega_i - (\Gamma_{ii+1} \cup \Gamma_{i-1i}) \times (0,T) \\
c_i(x,0) = c_0(x) & in\ \Omega_i \\
c_i = c_j & on\ \ \Gamma_{ij} \quad i, j = 1,...,q \\
\dfrac{\partial c_i}{\partial n} = \dfrac{\partial c_j}{\partial n} & on\ \ \Gamma_{ij}
\end{cases}
$$

The last two interface conditions are known as transmission conditions on Γ_{ij}.

The decomposed problem (3.1) is discretized on each sub-domain Ω_i, i=1,…,q using the implicit finite volume scheme described in Section 2. For the interface conditions we have used the centred differences scheme. We obtain the following system for i=1, …, q-1 and j=i+1:

$$
(3.2)\quad \begin{cases}
a_{P_i} c_{P_i}^{n+1} + a_{W_i} c_{W_i}^{n+1} + a_{N_i} c_{N_i}^{n+1} + a_{S_i} c_{S_i}^{n+1} = b_{P_i} & in\ \Omega_i \quad (a) \\
a_{P_j} c_{P_j}^{n+1} + a_{E_j} c_{E_j}^{n+1} + a_{N_j} c_{N_j}^{n+1} + a_{S_j} c_{S_j}^{n+1} = b_{P_j} & in\ \Omega_j \quad (b) \\
c_{e_i}^{n+1} = c_{w_j}^{n+1} & on\ \ \Gamma_{ij} \quad (c) \\
c_{e_i}^{n+1} + c_{e_j}^{n+1} - c_{p_i}^{n+1} - c_{p_j}^{n+1} = 0 & on\ \ \Gamma_{ij} \quad (d)
\end{cases}
$$

where

$$
\begin{cases}
\sigma_i = e_i \ and\ \sigma_j = e_j \quad if\ V_{P_i} \cap \Gamma_{ij} \neq \emptyset\ (i = 1, \dots, q-1) \\
\sigma_i = E_i \ and\ \sigma_j = W_j \quad else
\end{cases}
$$

b_{P_i} is a constant depending on, the source term $g_{P_i}^n$, $c_{P_i}^n$, the discritized convection flux, the boundary and the initial conditions in Ω_i, i=1,...,q.

Schur complement

The methods based on Schur Complement exists in two versions. The first one uses the Steklov Poincaré operator and the second one is an algebraic version.

For exemple in [1, 2, 4] and in [5] , one finds presentations of these methods (for linear advection diffusion equation) used in the context of a finite elements method and finite volumes method, respectively.

In this work, we have used an algebraic version of Schur Complement technique.

Let C_i^{n+1} and C_Γ^{n+1} denote the vector of the unknowns of Ω_i (i=1,…, q) and Γ at time t_{n+1} (respectively), and b_i denote the vector of b_{P_i}.

The decomposed problem (3.2) can be written in the following matrix form:

$$
(3.3)\quad \begin{bmatrix}
A_1 & 0 & \cdots & 0 & A_{1\Gamma} \\
0 & A_2 & \cdots & 0 & A_{2\Gamma} \\
. & . & \cdots & . & . \\
0 & \cdots & 0 & A_q & A_{q\Gamma} \\
A_{\Gamma 1} & A_{\Gamma 2} & \cdots & A_{\Gamma q} & A_{\Gamma\Gamma}
\end{bmatrix}
\begin{bmatrix}
C_1^{n+1} \\ C_2^{n+1} \\ . \\ C_q^{n+1} \\ C_\Gamma^{n+1}
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ . \\ b_q \\ 0
\end{bmatrix}
$$

with

A_i, $A_{i\Gamma}$ describe respectively (a) and (b) of system (3.2), and $A_{\Gamma i}$, $A_{\Gamma\Gamma}$ (i=1,…,q) describe respectively (c) and (d) of system (3.2).

The matrix A_i present the coupling of the unknowns in Ω_i, $A_{\Gamma\Gamma}$ it is related to the unknowns on the interface, $A_{\Gamma i}$ and $A_{i\Gamma}$ representing the coupling of the unknowns of each sub-domain Ω_i with those of the interface $\Gamma_{i,i+1}$ for (i=1,…, q-1).

The system (3.3) can be sought formally by block Gaussian elimination.

Eliminating C_i^{n+1} (i=1,…,q) in the system (3.3), yields the following reduced linear system for C_Γ^{n+1}:

$$(3.4) \qquad SC_\Gamma^{n+1} = \chi_\Gamma,$$

where

$$\chi_\Gamma = -\sum_{i=1,\dots,q} A_{\Gamma i} A_i^{-1} b_i,$$

and

$$S = A_{\Gamma\Gamma} - \sum_{i=1,\dots,q} A_{\Gamma i} A_i^{-1} A_{i\Gamma},$$

S is the Schur Complement matrix.

After calculating, C_Γ^{n+1}, C_i^{n+1} can be obtained immediately and independently (in parallel) by solving

$$(3.5) \qquad A_i C_i^{n+1} = b_i - A_{i\Gamma} C_\Gamma^{n+1} \text{ (i=1,…,q)}$$

Numerical Simulations

In this section, we shall verify the proposed approach by numerical experiments.

Let us apply FV mono-domain (FV-MonoD) and the combined FV method Schur Complement (FV-SC) to the *2D* viscous Burgers equation [6, 7, 8]:

$$(4.1) \qquad \frac{\partial c}{\partial t} - \nu \Delta c + c\frac{\partial c}{\partial x_1} + c\frac{\partial c}{\partial x_2} = \mathrm{g},$$

The spatial domain is the square $\Omega_\mathrm{i} = (-1,1)^2$, the time interval $T = (0,1)$, $\nu = 0.01$, the initial data $c_0 = 0$ and the Dirichlet conditions $c_\mathrm{D} = 0$. The right-hand side g is chosen so that it conforms to the exact solution [8]:

$$c(x,t) = (1 - e^{-2t})(1 - x_1^2)^2(1 - x_2^2)^2$$

As we want to examine the error of the space discretization, we overkill the time step so that the time discretization error is negligible.

Figure 4 (a,b,c,d) show respectively the analytical, the numerical mono-domain, the multi-domain (q=2) and the multi-domain (q=9) solutions.

Figure 5 shows the convergence of the proposed algorithm when varying the mesh of calculation.

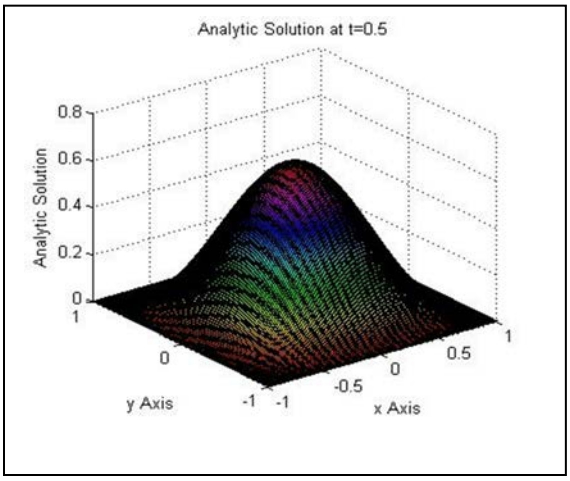

Figure 4a. Analytical solution

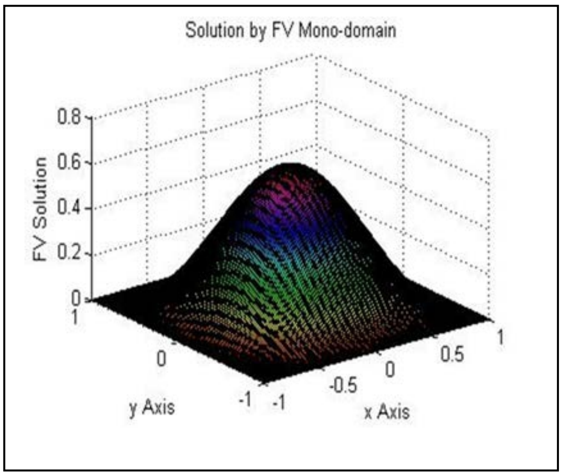

Figure 4b. Numerical mono-domain solution

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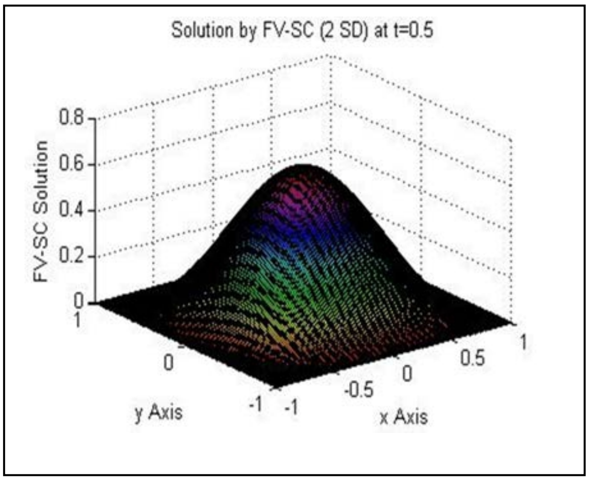

Figure 4c. Numerical multi-domain (q=2) solution

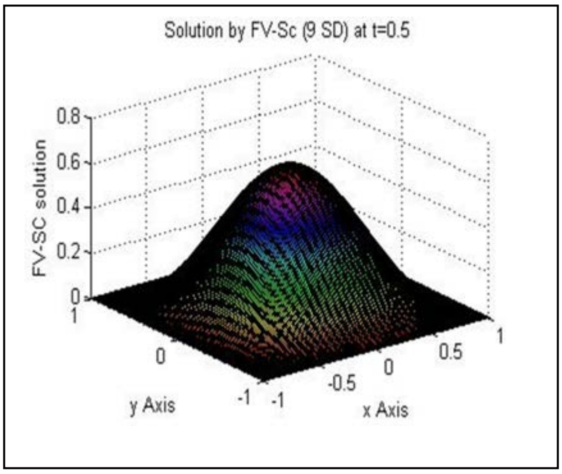

Figure 4d. Numerical multi-domain (q=9) solution

Figure 4. Numerical and analytical solution

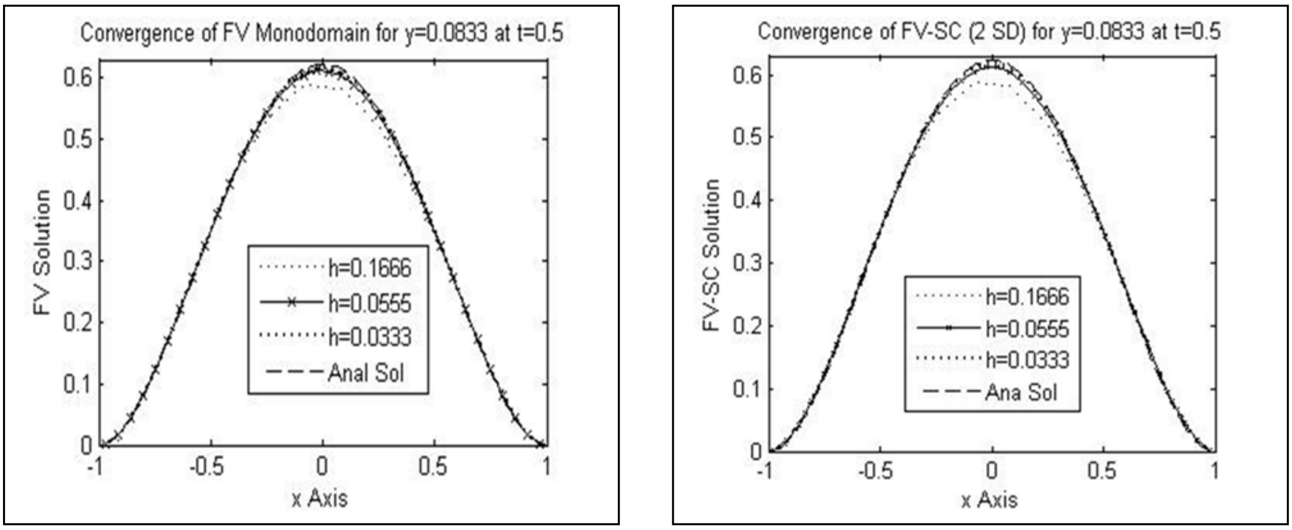

Figure 5. Convergence of numerical scheme

Conclusion

A new approach coupling implicit FV and Algebraic Schur Complement methods applied to a semi linear advection-diffusion equation, on 2D structured and conforming mesh, is presented.

The numerical experiments show that the proposed algorithm applied to a non-overlapping multi-subdomain decomposition has both the properties of stability and accuracy.

On the other hand , its reduces the calculation cost compared to global FV calculation.

As perspective of this work we project to develop a new algorithm integrating the non linear advection part implicitly. This algorithm will include for example Newton method to compute the advection term after each time step of the numerical scheme.

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