ON THE TOPOLOGICAL STRUCTURES ON Γ-SEMIRINGS

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Abstract:
In this paper, we introduce some special classes of ideals in Γ-semirings called prime *k*-ideal, prime full *k*-ideal, prime ideals, maximal and strongly irreducible ideals. Considering and investigating properties of the collection $\mathcal{A}$, $\mathcal{T}$, $\mathcal{M}$, $\mathcal{B}$ and $\mathcal{S}$ of all proper prime *k*-ideals, proper prime full *k*-ideals, maximal ideals, prime ideals and strongly irreducible ideals, respectively, of a Γ-semiring *R*, we construct the respective topologies on them by means of closure operator defined in terms of intersection and inclusion relation among these ideals of Γ-semiring *R*. The respective obtained topological spaces are called the structure spaces of the Γ-semiring *R*. We study a several principal topological axioms and properties in those structure spaces of Γ-semiring such as separation axioms, compactness and connectedness etc.

Key Words: Γ-Semiring; Prime *k*-ideal (ideal); (strongly) irreducible ideal; Hull-Kernel topology; Structure space

Introduction and preliminaries

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory etc.

The theory of semiring was first developed by H. S. Vandiver [33] and he has obtained important results of the objects. Semiring constitute a fairly natural generalization of rings, with board applications in the mathematical foundation of computer science. Also, semiring theory has many applications to other branches. For example, automata theory, optimization theory, algebra of formal process, combinatorial optimization, Baysian networks and belief propagation (cf. [12, 13, 14]).

It is well known that the concept of Γ-rings was first introduced and investigated by Nobusawa in 1964 [27], which is a generalization of the concept of rings. The class of Γ-rings contains not only all rings but also all Hestenes ternary rings. Later Barnes [2] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. After these two papers were published, many mathematicians obtained interesting results on Γ-rings in the sense of Barnes and Nobusawa extending and generalizing many classical notions and results of the theory of rings. Γ-semirings were first studied by M. K. Rao [28] as a generalization of Γ-ring as well as of semiring. The concepts of Γ-semirings and its sub-Γ-semirings with a left(right) unity was studied by J. Luh [26] and M. K. Rao in [28]. The ideals, prime ideals, semiprime ideals, k-ideals and h-ideals of a Γ-semiring, regular Γ-semiring, respectively, were extensively studied by S. Kyuno [21, 22, 24] (cf. [23]) and M. K. Rao [28, 29].

In Γ-semirings, the properties of their ideals, prime ideals, semiprime ideals and their generalizations play an important role in their structure theory, however the properties of an ideal in semirings and Γ-semirings are somewhat differerent from the properties of the usual ring ideals. In order to amend these differs, the concepts of *k*-ideals and *h*-ideals in a semiring were introduced and considered by D. R. LaTorre [25] in 1965. For the properties of some *h*-ideals in Γ-semirings, the reader is referred to the recent papers of T. K. Dutta and S. K. Sardar, K. P. Shum in [7,8,9, 10, 31].

The notion of Γ-semiring not only generalizes the notions of semiring and Γ-ring but also the notion of ternary semiring. We point out here that this notion provides an algebraic background to the non-positive cones of the totally ordered rings. We recall here that the non-negative cones of the totally ordered rings form semirings but the non-positive cones do not form semirings because the induced multiplication is no longer closed. For further study of semirings, Γ-semirings and their generalization and examples, the reader is referred to [7, 8, 9, 10, 16, 28, 29, 31].

In this paper, we introduce some special classes of ideals in Γ-semirings called prime *k*-ideal, prime full *k*-ideal, prime ideals, maximal and strongly irreducible ideals. Considering and investigating properties of the collection A, T, M, B and S of all proper prime *k*-ideals, proper prime full *k*-ideals, maximal ideals, prime ideals and strongly irreducible ideals, respectively, of a Γ-semiring R, we construct the respective topologies on them by means of closure operator defined in terms of intersection and inclusion relation among these ideals of Γ-semiring R. The respective obtained topological spaces are called the structure spaces of the Γ-semiring R. In fact we define this topology on A and topology on T will be the subspace topology from A since T is a subset of A. This topological space has been studied in different algebraic structures [1, 3, 4, 5, 6, 11, 18, 19, 20,32]. Recently, in [17] we have studied the topological structure on semihypergroups. We study several principal topological axioms and properties in those structure spaces of Γ-semiring such as separation axioms, compactness and connectedness etc.

Recall first the basic terms and definitions from the Γ-semiring theory.

Let *R* and Γ be two additive commutative semigroups. Then *R* is called a Γ-*semiring* if there exists a mapping $R\times\Gamma\times R\to R$ (the image to be denoted by $a\alpha b$, for $a,b\in R$ and $\alpha\in\Gamma$) satisfying the following conditions:

1. $a\alpha(b+c)=a\alpha b+a\alpha c$;
2. $(a+b)\alpha c=a\alpha c+b\alpha c$;
3. $a(\alpha+\beta)c=a\alpha c+a\beta c$;
4. $a\alpha(b\beta c)=(a\alpha b)\beta c$ for all $a,b,c\in R$ and $\alpha,\beta\in\Gamma$.

Obviously, every semiring R is a Γ-semiring with $\Gamma=R$ where $a\alpha b$ denotes the product of elements $a;\alpha;b\in R$, but not conversely.

If R contains an element 0 such that $0+x=x=x+0$ and $0\alpha x=x\alpha 0=0$ for all $x\in R$, for all $\alpha\in\Gamma$, then 0 is called the zero element (absorbing zero) or simply the zero of the Γ-semiring R. A non-empty subset T of R is said to be a *sub*-Γ-*semiring* of R if $(T,+)$ is a subsemigroup of $(R,+)$ and $a\alpha b\in T$; for all $a;b\in T$ and for all $\alpha\in\Gamma$. A non-empty subset I of a Γ-semiring R is called an *ideal* of R if $I+I\subseteq I, I\Gamma R\subseteq I, R\Gamma I\subseteq I$, where for subsets $U;V$ of R and Θ of Γ,

$$U\Theta V=\{\sum_{i=1}^{n}u_i\gamma_i v_i : u_i\in U, v_i\in V, \gamma_i\in\Theta \text{ and } n \text{ is a positive integer}\}.$$

An ideal I of a Γ-semiring R is called a k-*ideal* if for $x;y\in R; x+y\in I$ and $y\in I$ implies that $x\in I$. For a Γ-semiring R, let $E^+(R)=\{x\in R\mid x=x+x\}$. A k-ideal I of R is said to be full if $E^+(R)\subseteq I$. A proper ideal P of a Γ-semiring R is called a *prime ideal* of R if $a\Gamma b\subseteq P$ implies $a\in P$ or $b\in P$ for all a,b of R. An ideal I of a Γ-semiring R is called *proper* iff $I\subset R$ holds, where $\subset$ denotes proper inclusion, and a proper ideal I is called *maximal* iff there is no ideal A of R satisfying $I\subset A\subset R$. An element e of a Γ-semiring R is called *identity element* of R if $e\alpha x=x=x\alpha e$, for all $x\in R;\alpha\in\Gamma$.

Throughout this paper, *R* will always denote a Γ-semiring with zero and unless otherwise stated a Γ-semiring means a Γ-semiring with zero.

On topological space of prime k-hyperideals of Γ-semiring

Let we denote with A the collection of all prime k-ideals and T the collection of all prime full k-ideals of a Γ-semiring R. For any subset A of A, we define

$$\overline{A} = \{I \in \mathsf{A} : \bigcap_{I_i \in A} I_i \subseteq I\}.$$

It can be easily seen that $\overline{\varnothing} = \varnothing$.

Theorem 2.1 *Let* A, B *be any two subsets of* A *. Then*

1. $A \subseteq \overline{A}$.
2. $\overline{\overline{A}} = \overline{A}$.
3. $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$.
4. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. (1). It is clear that $\bigcap_{I_i \in A} I_i \subseteq I_i$ for each i and hence $A \subseteq \overline{A}$.

(2). By (1), we have $\overline{A} \subseteq \overline{\overline{A}}$. Conversely, let $I_j \in \overline{\overline{A}}$. Then $\bigcap_{I_i \in \overline{A}} I_i \subseteq I_j$. Now $I_i \in \overline{A}$ implies that $\bigcap_{I_t \in A} I_t \subseteq I_i$ for all $i \in \Lambda$. Thus

$$\bigcap_{I_t \in A} I_t \subseteq \bigcap_{I_i \in \overline{A}} I_i \subseteq I_j$$

So $I_j \in \overline{A}$ and hence $\overline{\overline{A}} \subseteq \overline{A}$. Consequently, $\overline{\overline{A}} = \overline{A}$.

(3). Let us suppose that $A \subseteq B$. Let $I_i \in \overline{A}$. Then $\bigcap_{I_j \in A} I_j \subseteq I_i$. Since $A \subseteq B$, it follows that

$$\bigcap_{I_j \in A} I_j \subseteq \bigcap_{I_j \in A} I_j \subseteq I_i.$$

This implies that $I_i \in \overline{B}$ and hence $\overline{A} \subseteq \overline{B}$.

(4). It is clear that $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Conversely, let $I_i \in \overline{A \cup B}$. Then $\bigcap_{I_j \in A \cup B} I_j \subseteq I_i$. It can be easily seen that

$$\bigcap_{I_j \in A \cup B} I_j = \left(\bigcap_{I_j \in A} I_j\right) \cap \left(\bigcap_{I_j \in B} I_j\right).$$

Since $\bigcap_{I_j \in A} I_j$ and $\bigcap_{I_j \in B} I_j$ are ideals of R, we have

$$\left(\bigcap_{I_j \in A} I_j\right) \Gamma \left(\bigcap_{I_j \in B} I_j\right) \subseteq \left(\bigcap_{I_j \in A} I_j\right) \cap \left(\bigcap_{I_j \in B} I_j\right) = \bigcap_{I_j \in A \cup B} I_j \subseteq I_i.$$

We have I_i is a prime ideal of R and hence either $\bigcap_{I_j \in A} I_j \subseteq I_i$ or $\bigcap_{I_j \in B} I_j \subseteq I_i$ i.e. either $I_i \in \overline{A}$ or $I_i \in \overline{B}$ i.e. $I_i \in \overline{A} \cup \overline{B}$. Consequently, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$ and hence $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Definition 2.2 *The closure operator* $A \to \overline{A}$ *gives a topology* τ_{A} *on* A *. This topology* τ_{A} *is called the hull-kernel topology and the topological space* $(\mathsf{A}, \tau_{\mathsf{A}})$ *is called the structure space of the* Γ-*semiring* R.

Let I be a k-ideal of a Γ-semiring R. We define

$$\Delta(I) = \{I' \in \mathsf{A} : I \subseteq I'\} \text{ and } C\Delta(I) = \mathsf{A} \setminus \Delta(I) = \{I' \in \mathsf{A} : I \acute{U} I'\}.$$

Proposition 2.3 *Let* R *be a* Γ-*semiring and* I *a* k-*ideal of* R*. Then any closed set in* A *is of the form* $\Delta(I)$.

Proof. Let $\overline{A}$ be any closed set in $\mathcal{A}$, where $A \subseteq \mathcal{A}$. Let $A = \{I_i : i \in \Lambda\}$ and $I = \bigcap_{I_i \in A} I_i$. Then I is a k-ideal of R. Let $I' \in \overline{A}$. Then $\bigcap_{I_i \in A} I_i \subseteq I'$. This implies that $I \subseteq I'$. Consequently, $I' \subseteq \Delta(I)$. So $\overline{A} \subseteq \Delta(I)$.

Conversely, let $I' \in \Delta(I)$. Then $I \subseteq I'$ i.e. $\bigcap_{I_i \in A} I_i \subseteq I'$. Consquently, $I' \in \overline{A}$ and hence $\Delta(I) \subseteq \overline{A}$. Thus $\overline{A} = \Delta(I)$.

Corollary 2.4 *Let R be a Γ-semiring and I a k-ideal of R. Then any open set in $\mathcal{A}$ is of the form $C\Delta(I)$.*

Let R be a Γ-semiring and $a \in R$. We define

$$\Delta(a) = \{I \in \mathcal{A} : a \in I\} \text{ and } C\Delta(a) = \mathcal{A} \setminus \Delta(a) = \{I \in \mathcal{A} : a \notin I\}.$$

Proposition 2.5 *Let R be a Γ-semiring and $a \in R$. Then $\{C\Delta(a) : a \in R\}$ forms an open base for the hull-kernel topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$.*

Proof. Let $U \in \tau_{\mathcal{A}}$. Then $U = C\Delta(I)$, where I is a k-ideal of R. Let $J \in U = C\Delta(I)$. Then $I \not\subseteq J$. This implies that there exists $a \in I$ such that $a \notin J$. Thus $J \in C\Delta(a)$. It remains to show that $C\Delta(a) \subset U$. Let $K \in C\Delta(a)$. Then $a \notin K$. This implies that $I \not\subseteq K$. Consequently, $K \in U$ and hence $C\Delta(a) \subset U$. So we find that $J \in C\Delta(a) \subset U$. Thus $\{C\Delta(a) : a \in R\}$ is an open base for the hull-kernel topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$.

Theorem 2.6 *Let R be a Γ-semiring. The structure space $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_0-space.*

Proof. Let I_1 and I_2 be two distinct elements of $\mathcal{A}$. Then there is an element a either in $I_1 \setminus I_2$ or in $I_2 \setminus I_1$. Let us suppose that $a \in I_1 \setminus I_2$. Then $C\Delta(a)$ is a neighbourhood of I_2 not containing I_1. Hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_0-space.

Theorem 2.7 *Let R be a Γ-semiring. $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_1-space if and only if no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$.*

Proof. Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a T_1-space. Let us suppose that I_1 and I_2 be any two distinct elements of $\mathcal{A}$. Then each of I_1 and I_2 has a neighbourhood not containing the other. Since I_1 and I_2 are arbitrary elements of $\mathcal{A}$, it follows that no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$.

Conversely, let us suppose that no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$. Let I_1 and I_2 be any two distinct elements of $\mathcal{A}$. Then by hypothesis, $I_1 \not\subset I_2$ and $I_2 \not\subset I_1$. This implies that there exist $a, b \in R$ such that $a \in I_1$, but $a \notin I_2$ and $b \in I_2$, but $b \notin I_1$. Consequently, we have $I_1 \in C\Delta(b)$, but $I_1 \notin C\Delta(a)$ and $I_2 \in C\Delta(a)$, but $I_2 \notin C\Delta(b)$ i.e. each of I_1 and I_2 has a neighbourhood not containing the other. Hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is a T_1-space.

Corollary 2.8 *Let $\mathcal{M}$ be the set of all proper maximal k-ideals of a Γ-semiring R with identity. Then $(\mathcal{M}, \tau_{\mathcal{M}})$ is a T_1-space, where $\tau_{\mathcal{M}}$ is the induced topology on $\mathcal{M}$ from $(\mathcal{A}, \tau_{\mathcal{A}})$.*

Theorem 2.9 *Let R be a Γ-semiring. $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorff space if and only if for any two distinct pair of elements I, J of $\mathcal{A}$, there exist $a, b \in R$ such that $a \notin I, b \notin J$ and there does not exist any element K of $\mathcal{A}$ such that $a \notin K$ and $b \notin K$.*

Proof. Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a Hausdorff space. Then for any two distinct elements I, J of $\mathcal{A}$, there exist basic open sets $C\Delta(a)$ and $C\Delta(b)$ such that $I \in C\Delta(a)$, $J \in C\Delta(b)$ and $C\Delta(a) \cap C\Delta(b) = \varnothing$. Now $I \in C\Delta(a)$ and $J \in C\Delta(b)$ imply that $a \notin I$ and $b \notin J$. If possible, let $K \in \mathcal{A}$ such that $a \notin K$ and $b \notin K$. Then $K \in C\Delta(a)$ and hence $K \in C\Delta(a) \cap C\Delta(b)$. It is impossible, since $C\Delta(a)\cap C\Delta(b)=\varnothing$. Thus there does not exist any element $K\in \mathsf{A}$ such that $a\notin K$ and $b\notin K$.

Conversely, let us suppose that the given condition holds and $I,J\in \mathsf{A}$ such that $I\neq J$. Let $a,b\in R$ be such that $a\notin I, b\notin J$ and there does not exist any $K\in \mathsf{A}$ such that $a\notin K$ and $b\notin K$. Then $I\in C\Delta(a), J\in C\Delta(b)$ and $C\Delta(a)\cap C\Delta(b)=\varnothing$. This implies that $(\mathsf{A},\tau_{\mathsf{A}})$ is a Hausdorff space.

Corollary 2.10 *Let R be a Γ-semiring. If $(\mathsf{A},\tau_{\mathsf{A}})$ is a Hausdorff space, then no proper prime k-ideal contains any other proper prime k-ideal. If $(\mathsf{A},\tau_{\mathsf{A}})$ contains more than one element, then there exist $a,b\in R$ such that $\mathsf{A}=C\Delta(a)\cup C\Delta(b)\cup\Delta(I)$, where I is the k-ideal generated by a,b.*

Proof. Let us suppose that ($(\mathsf{A},\tau_{\mathsf{A}})$ is a Hausdorff space. Since every Hausdorff space is a T_1-space, $(\mathsf{A},\tau_{\mathsf{A}})$ is a T_1-space. Hence by Theorem 2.7, it follows that no proper prime k-ideal contains any other proper prime k-ideal. Now let $J,K\in \mathsf{A}$ be such that $J\neq K$. Since $(\mathsf{A},\tau_{\mathsf{A}})$ is a Hausdorff space, there exist basic open sets $C\Delta(a)$ and $C\Delta(b)$ such that $J\in C\Delta(a), K\in C\Delta(b)$ and $C\Delta(a)\cap C\Delta(b)=\varnothing$. Let I be the k-ideal generated by a,b. Then I is the smallest k-ideal containing a and b. Let $K\in \mathsf{A}$. Then either $a\in K, b\notin K$ or $a\notin K, b\in K$ or $a,b\in K$. The case $a\notin K, b\notin K$ is not possible, since $C\Delta(a)\cap C\Delta(b)=\varnothing$. Now in the first case, $K\in C\Delta(a)$ and hence $\mathsf{A}\subseteq C\Delta(a)\cup C\Delta(b)\cup\Delta(I)$. In the second case, $K\in C\Delta(a)$ and hence $\mathsf{A}\subseteq C\Delta(a)\cup C\Delta(b)\cup\Delta(I)$. In the third case, $K\in\Delta(I)$ and hence $\mathsf{A}\subseteq C\Delta(a)\cup C\Delta(b)\cup\Delta(I)$. So we find that $\mathsf{A}\subseteq C\Delta(a)\cup C\Delta(b)\cup\Delta(I)$. Again, clearly $C\Delta(a)\cup C\Delta(b)\cup\Delta(I)\subseteq \mathsf{A}$. Hence $\mathsf{A}=C\Delta(a)\cup C\Delta(b)\cup\Delta(I)$.

Theorem 2.11 *Let R be a Γ-semiring. $(\mathsf{A},\tau_{\mathsf{A}})$ is a regular space if and only if for any $I\in \mathsf{A}$ and $a\notin I, a\in R$, there exists a k-ideal J of R and $b\in R$ such that $I\in C\Delta(b)\subseteq\Delta(J)\subseteq C\Delta(a)$.*

Proof. Let $(\mathsf{A},\tau_{\mathsf{A}})$ be a regular space. Let $I\in \mathsf{A}$ and $a\notin I$. Then $I\in C\Delta(a)$ and $\mathsf{A}\setminus C\Delta(a)$ is a closed set not containing I. Since $(\mathsf{A},\tau_{\mathsf{A}})$ is a regular space, there exist disjoints open sets U and V such that $I\in U$ and $\mathsf{A}\setminus C\Delta(a)\subseteq V$. This implies that $\mathsf{A}\setminus V\subseteq C\Delta(a)$. Since V is open, $\mathsf{A}\setminus V$ is closed and hence there exists a k-ideal J of R such that $\mathsf{A}\setminus V=\Delta(J)$, by Proposition 2.3. So we find that $\Delta(J)\subseteq C\Delta(a)$. Again, since $U\cap V=\varnothing$, we have $V\subseteq \mathsf{A}\setminus U$. Since U is open, $\mathsf{A}\setminus U$ is closed and hence there exists a k-ideal K of R such that $\mathsf{A}\setminus U=\Delta(K)$ i.e. $V\subseteq\Delta(K)$. Since $I\in U, I\notin \mathsf{A}\setminus U=\Delta(K)$. This implies that KÚI. Thus there exists $b\in K(\subset R)$ such that $b\notin I$. So $I\in C\Delta(b)$. Now we show that $V\subseteq\Delta(b)$. Let $M\in V\subseteq\Delta(K)$. Then $K\subseteq M$. Since $b\in K$, it follows that $b\in M$ and hence $M\in\Delta(b)$. Consequently, $V\subseteq\Delta(b)$. This implies that $\mathsf{A}\setminus\Delta(b)\subseteq \mathsf{A}\setminus V=\Delta(J)\Rightarrow C\Delta(b)\subseteq\Delta(J)$. Thus we find that $I\in C\Delta(b)\subseteq\Delta(J)\subseteq C\Delta(a)$.

Conversely, let us suppose that the given condition holds. Let $I\in \mathsf{A}$ and $\Delta(K)$ be any closed set not containing I. Since $I\notin\Delta(K)$, we have $K\not\subset I$. This implies that there exists $a\in K$ such that $a\notin I$. Now by the given condition, there exists a k-ideal J of R and $b\in R$ such that $I\in C\Delta(b)\subseteq\Delta(J)\subseteq C\Delta(a)$. Since $a\in K, C\Delta(a)\cap\Delta(K)=\varnothing$. This implies that $\Delta(K)\subseteq \mathsf{A}\setminus C\Delta(a)\subseteq \mathsf{A}\setminus\Delta(J)$. Since $\Delta(J)$ is a closed set, $\mathsf{A}\setminus\Delta(J)$ is an open set containing the closed set $\Delta(K)$. Clearly, $C\Delta(b)\cap(\mathsf{A}\setminus\Delta(J))=\varnothing$. So we find that $C\Delta(b)$ and $\mathsf{A}\setminus(J)$ are two disjoints open sets containing I and $\Delta(K)$ respectively. Consequently, $(\mathsf{A},\tau_{\mathsf{A}})$ is a regular space.

Theorem 2.12 *Let R be a Γ-semiring. $(\mathsf{A}, \tau_{\mathsf{A}})$ is a compact space if and only if for any collection $\{a_\alpha\}_{\alpha\in\Lambda} \subset R$, there exists a finite subcollection $\{a_i : i = 1,2,...,n\}$ in R such that for any $I \in \mathsf{A}$, there exists a_i such that $a_i \notin I$.*

Proof. Let $(\mathsf{A}, \tau_{\mathsf{A}})$ be a compact space. Then the open cover $\{C\Delta(a_i) : a_i \in R\}$ of $(\mathsf{A}, \tau_{\mathsf{A}})$ has a finite subcover $\{C\Delta(a_i) : i = 1,2,...,n\}$. Let $I \in \mathsf{A}$. Then $I \in C\Delta(a_i)$ for some $a_i \in R$. This implies that $a_i \notin I$. Hence $\{a_i : i = 1,2,...,n\}$ is the required finite subcollection of elements of R such that for any $I \in \mathsf{A}$, there exists a_i such that $a_i \notin I$.

Conversely, let us suppose that the given condition holds. Let $\{C\Delta(a_i) : a_i \in R\}$ be an open cover of A. Suppose to the contrary that no finite subcollection of $\{C\Delta(a_i) : a_i \in R\}$ covers A. This means that for any finite set $\{a_1, a_2, ..., a_n\}$ of elements of R,

$$C\Delta(a_1) \cup C\Delta(a_2) \cup ... \cup C\Delta(a_n) \neq \mathsf{A} \Rightarrow$$
$$\Rightarrow \Delta(a_1) \cap \Delta(a_2) \cap ... \cap \Delta(a_n) \neq \varnothing \Rightarrow$$
$$\Rightarrow \text{there exists } I \in \mathsf{A} \text{ such that } I \in \Delta(a_1) \cap \Delta(a_2) \cap ... \cap \Delta(a_n) \Rightarrow$$
$$\Rightarrow a_1, a_2, ..., a_n \in I, \text{which contradicts our hypothesis.}$$

So the open cover $\{C\Delta(a_i) : a_i \in R\}$ has a finite subcover and hence $(\mathsf{A}, \tau_{\mathsf{A}})$ is compact.

Corollary 2.13 *If the Γ-semiring R is finitely generated, then $(\mathsf{A}, \tau_{\mathsf{A}})$ is a compact space.*

Proof. Let $\{a_i : i = 1,2,...,n\}$ be a finite set of generators of R. Then for any $I \in \mathsf{A}$, there exists a_i such that $a_i \notin I$, since I is a proper prime k-ideal of R. Hence by Theorem 2.12, $(\mathsf{A}, \tau_{\mathsf{A}})$ is a compact space.

Proposition 2.14 *Let R be a Γ-semiring. $(\mathsf{T}, \tau_{\mathsf{A}})$ is compact space if $E^+(R) \neq \{0\}$.*

Proof. Let $\{\Delta(I_i) \mid i \in \Lambda\}$ be any collection of closed sets in T with finite intersection property. Let I be the proper prime k-ideal which is also full k-ideal generated by $E^+(R)$. Since any prime, full k-ideal J of R contains $E^+(R)$, then J contains I. Hence $I \in \bigcap_{i\in\Lambda} \Delta(I_i) \neq \varnothing$. Consequently, $(\mathsf{T}, \tau_{\mathsf{T}})$ is compact.

Definition 2.15 *A Γ-semiring R is called a k-Noetherian Γ-semiring if it satisfies the ascending chain condition on k-ideals of R i.e. if $I_1 \subseteq I_2 \subseteq ... \subseteq I_n \subseteq ...$ is an ascending chain of k-ideals of R, then there exists a positive integer m such that $I_n = I_m$ for all $n \geq m$.*

Theorem 2.16 *If R is a k-Noetherian Γ-semiring, then $(\mathsf{A}, \tau_{\mathsf{A}})$ is countably compact.*

Proof. Let $\{\Delta(I_n\}_{n=1}^{\infty}$ be a countable collection of closed sets in A with finite intersection property (FIP). Let us consider the following ascending chain of prime k-ideals of R: $< I_1 > \subseteq < I_1 \cup I_2 > \subseteq < I_1 \cup I_2 \cup I_3 > \subseteq ....$

Since R is a k-Noetherian Γ-semiring, there exists a positive integer m such that $< I_1 \cup I_2 \cup ... I_m > = < I_1 \cup I_2 \cup ... I_{m+1} > = ...$ Thus it follows that $< I_1 \cup I_2 \cup ... I_m > \in \bigcap_{n=1}^{\infty} \Delta(I_n)$. Consequently, $\bigcap_{n=1}^{\infty} \Delta(I_n) \neq \varnothing$ and hence $(\mathsf{A}, \tau_{\mathsf{A}})$ is countably compact.

Corollary 2.17 *If R is a k-Noetherian Γ-semiring and $(\mathsf{A}, \tau_{\mathsf{A}})$ is second countable, then $(\mathsf{A}, \tau_{\mathsf{A}})$ is compact.*

Proof. The proof follows by Theorem 2.16 and the fact that a second countable space is compact if it is countably compact.

Remark 2.18 *Let $\{I_i\}$ be a collection of prime k-ideals of a Γ-semiring R. Then $\bigcap I_i$ is a k-ideal of R but it may not be a prime k-ideal of R, in general.*

For this we have the following proposition:

Proposition 2.19 *Let R be a Γ-semiring and $\{I_i\}$ be a collection of prime k-ideals of R such that $\{I_i\}$ forms a chain. Then $\bigcap I_i$ is a prime k-ideal of R.*

Proof. It is clear that $\bigcap I_i$ is a k-ideal of R. Let $A\Gamma B) \subseteq \bigcap I_i$ for any two k-ideals A, B of R. If possible, let $A, B \nsubseteq \bigcap I_i$. Then there exist i, j such that $A \nsubseteq I_i$, $B \nsubseteq I_j$. Since I_i is a chain, let $I_i \subseteq I_j$. This implies that $B \nsubseteq I_i$. Since $A\Gamma B \subseteq I_i$ and I_i is prime, we must have either $A \subseteq I_i$ or $B \subseteq I_i$. It is impossible. Therefore, either $A \subseteq \bigcap I_i$ or $B \subseteq \bigcap I_i$. Consequently, $\bigcap I_i$ is a prime k-ideal of R.

Theorem 2.20 *Let R be a Γ-semiring. $(\mathsf{A}, \tau_{\mathsf{A}})$ is disconnected if and if there exist a k-ideal I of R and a collection of points $\{a_i\}_{i\in\Lambda}$ of R not belonging to I such that if $I' \in \mathsf{A}$ and $a_i \in I', \forall i \in \Lambda$, then $I \setminus I' \neq \emptyset$.*

Proof. Let $(\mathsf{A}, \tau_{\mathsf{A}})$ be not connected. Then there exists a non-trivial open and closed subset of A. Let I be the k-ideal of R for which $\Delta(I)$ is closed as well as open. Then $\Delta(I) = \bigcup_{i\in\Lambda} C\Delta(a_i)$ where $\{a_i\}_{i\in\Lambda}$ is a collection of points of R. Now since $C\Delta(a_i) \subseteq \Delta(I), \forall i \in \Lambda$ for any $I_i \in C\Delta(a_i)$ we have $I \subseteq I_i$, therefore $a_i \notin I$ as $a_i \notin I_i, \forall i \in \Lambda$. For any $I' \in \mathsf{A}$ and $a_\alpha \in I', \forall i \in \Lambda$ we have $I' \notin \Delta(I)$, consequently $I \nsubseteq I'$, i.e. $I \setminus I' \neq \emptyset$.

Conversely, let us suppose the the given condition holds. Then $\Delta(I) = \bigcup_{i\in\Lambda} C\Delta(a_i)$ is an open and closed non-trivial subset of A and hence $(\mathsf{A}, \tau_{\mathsf{A}})$ is disconnected.

Definition 2.21 *Let R be a Γ-semiring. The structure space $(\mathsf{A}, \tau_{\mathsf{A}})$ of R is called irreducible if for any decomposition $\mathsf{A} = \mathsf{A}_1 \cup \mathsf{A}_2$, where A_1, A_2 are closed subsets of A, we have either $\mathsf{A} = \mathsf{A}_1$ or $\mathsf{A} = \mathsf{A}_2$.*

Theorem 2.22 *Let R be a Γ-semiring and A be a closed subset of A. Then A is irreducible if and only if $\bigcap_{I_i\in A} I_i$ is a prime k-ideal of R.*

Proof. Let A be irreducible. Let P, Q be two k-ideals of R such that $P\Gamma Q \subseteq \bigcap_{I_i\in A} I_i$. Then $P\Gamma Q \subseteq I_i$ for all i. Since I_i is prime, we have $P \subseteq I_i$ or $Q \subseteq I_i$ which implies for $I_i \in A$, $I_i \in \{\overline{P}\}$ or $I_i \in \{\overline{Q}\}$. Hence $A = (A \cap \overline{P}) \cup (A \cap \overline{Q})$. Since A is irreducible and $(A \cap \overline{P}), (A \cap \overline{Q})$, are closed, it follows that $A = A \cap \overline{P}$ or $A = A \cap \overline{Q}$ and hence $A \subseteq \overline{P}$ or $A \subseteq \overline{Q}$. This implies that $P \subseteq \bigcap_{I_i\in A} I_i$ or $Q \subseteq \bigcap_{I_i\in A} I_i$. Consequently, $\bigcap_{I_i\in A} I_i$ is a prime k-ideal of R.

Conversely, let us suppose that $\bigcap_{I_i\in A} I_i$ is a prime k-ideal of R. Let $A = A_1 \cup A_2$, where A_1, A_2 are closed subsets of A. Then $\bigcap_{I_i\in A} I_i \subseteq \bigcap_{I_i\in A_1} I_i$, $\bigcap_{I_i\in A} I_i \subseteq \bigcap_{I_i\in A_2} I_i$. We have

$$\bigcap_{I_i\in A} I_i = \bigcap_{I_i\in A_1\cup A_2} I_i = \left(\bigcap_{I_i\in A_1} I_i\right) \cap \left(\bigcap_{I_i\in A_2}\right).$$

Also

$$\left(\bigcap_{I_i \in A_1} I_i\right)\Gamma\left(\bigcap_{I_i \in A_2} I_i\right) \subseteq \left(\bigcap_{I_i \in A_1} I_i\right), \left(\bigcap_{I_i \in A_1} I_i\right)\Gamma\left(\bigcap_{I_i \in A_2} I_i\right) \subseteq \left(\bigcap_{I_i \in A_2} I_i\right).$$

Thus we have

$$\left(\bigcap_{I_i \in A_1} I_i\right)\Gamma\left(\bigcap_{I_i \in A_2} I_i\right) \subseteq \left(\bigcap_{I_i \in A_1} I_i\right) \cap \left(\bigcap_{I_i \in A_2}\right).$$

Since $\bigcap_{I_i \in A} I_i$ is prime, it follows that

$$\bigcap_{I_i \in A_1} I_i \subseteq \bigcap_{I_i \in A} I_i \text{ or } \bigcap_{I_i \in A_2} I_i \subseteq \bigcap_{I_i \in A} I_i .$$

So we find that

$$\bigcap_{I_i \in A} I_i = \bigcap_{I_i \in A_1} I_i \text{ or } \bigcap_{I_i \in A} I_i = \bigcap_{I_i \in A_2} I_i .$$

Let $I_\beta \in A$. We have

$$\bigcap_{I_i \in A_1} I_i \subseteq I_\beta \text{ or } \bigcap_{I_i \in A_2} I_i \subseteq I_\beta .$$

Since $A_1, A_2 \subseteq A$, so $I_i \subseteq I_\beta$ for all $I_i \in A_1$ or $I_i \subseteq I_\beta$ for all $I_i \in A_2$. Thus $I_\beta \in \overline{A_1} = A_1$ or $I_\beta \in \overline{A_2} = A_2$, since A_1 or A_2 are closed, i.e. $A = A_1$ or A_2.

Theorem 2.23 *Let R be a Γ-semiring. $(\mathcal{A}, \tau_{\mathcal{A}})$ is disconnected if and if there exist a k-ideal I of R and a collection of points $\{a_i\}_{i\in\Lambda}$ of R not belonging to I such that if $I' \in \mathcal{A}$ and $a_i \in I', \forall i \in \Lambda$, then $I \setminus I' \neq \varnothing$.*

Proof. Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be not connected. Then there exists a non-trivial open and closed subset of $\mathcal{A}$. Let I be the k-ideal of R for which $\Delta(I)$ is closed as well as open. Then $\Delta(I) = \bigcup_{i\in\Lambda} C\Delta(a_i)$ where $\{a_i\}_{i\in\Lambda}$ is a collection of points of R. Now since $C\Delta(a_i) \subseteq \Delta(I), \forall i \in \Lambda$ for any $I_i \in C\Delta(a_i)$ we have $I \subseteq I_i$, therefore $a_i \notin I$ as $a_i \notin I_i, \forall i \in \Lambda$. For any $I' \in \mathcal{A}$ and $a_i \in I', \forall i \in \Lambda$ we have $I' \notin \Delta(I)$, consequently IÚI', i.e. $I \setminus I' \neq \varnothing$.

Conversely, let us suppose the the given condition holds. Then $\Delta(I) = \bigcup_{i\in\Lambda} C\Delta(a_i)$ is an open and closed non-trivial subset of $\mathcal{A}$ and hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is disconnected.

Proposition 2.24 *Let R be a Γ-semiring. $(\mathcal{T}, \tau_{\mathcal{A}})$ is connected space if $E^+(R) \neq \{0\}$.*

Proof. Let I be the proper prime k-ideal generated by $E^+(R)$. Since any full k-ideal of R contains $E^+(R)$, then I belongs to any closed set $\Delta(I')$ of $\mathcal{A}$. Consequently, any two closed sets of $\mathcal{A}$ are not disjoint. Hence $(\mathcal{T}, \tau_{\mathcal{T}})$ is connected

On topological space of maximal ideals of Γ-semiring

In this section, the structure space of all maximal ideals of a Γ-semiring R with identity e is considered and studied.

An ideal is maximal if there is no ideal containing properly it. Let $\mathcal{M}$ be the set of all maximal ideals in a Γ-semiring R. We shall define two topologies on $\mathcal{M}$. For every $x \in R$, we denote by Δ_x the set of all maximal ideals containing x, by Ω_x the set $\mathcal{M} - \Delta_x$, i.e. the set of all maximal ideals not containing x. Let I be an ideal of R, we denote by Δ_I the set of all maximal ideals containing I.

We choose the family $\{\Delta_x \mid x \in R\}$ as a subbase for open sets of $\mathcal{M}$. We shall refer to the resulting topology on $\mathcal{M}$ as Δ-topology (in symbol, $\mathcal{M}_\Delta$). Similarly, we shall take the family $\{\Omega_x \mid x \in R\}$ as a subbase for open sets of $\mathcal{M}$ (in symbol, $\mathcal{M}_\Omega$).

Let M_1, M_2 be two distinct elements of $\mathcal{M}_\Delta$. Then we have $M_1 + M_2 = R$. Therefore there are a, b such that $e = a + b$ and $a \in M_1, b \in M_2$, so we have $\Delta_a \ni M_1, \Delta_b \ni M_2$ and $\Delta_a \cap \Delta_b = \varnothing$. Hence we have

Theorem 3.1 *The topological space $\mathcal{M}_\Delta$ is a T_2-space.*

Let now M be an element of $\mathcal{M}_\Gamma$, and $M \neq M_1 \in \mathcal{M}_\Omega$, then there is an element a such that $a \in M_1$ and $a \notin M$. Therefore $M_1 \notin \Omega_a$ and $M_1 \notin \bigcap_{x \notin M} \Omega_x$. This implies $M = \bigcap_{x \notin M} \Omega_x$. Hence we obtain the following

Theorem 3.2 *The topological space $\mathcal{M}_\Omega$ is a T_1-space.*

Let I be an ideal of R and $\{a_\lambda\}$ a generator of I, then we have

$$\Delta_I = \bigcap_\lambda \Delta_{a_\lambda}.$$

Therefore, the closed sets for the topological space $\mathcal{M}_\Omega$ have the form $\Delta_{I_1} \cup \Delta_{I_2} \cup \ldots \cup \Delta_{I_n}$, where I_i are ideals of R. Let $I = \bigcap_{i=1}^{n} I_i$, if $M \in \Delta_{I_i}$ for some i, then $M \supset I_i$ and $M \supset I$. This implies $\Delta_I \ni M$ and we have $\bigcup_{i=1}^{n} \Delta_{I_i} \subset \Delta_I$. Let us suppose that there is a maximal ideal M such that $M \in \Delta I - \bigcup_{i=1}^{n} \Delta_{I_i}$, then $M \in \Delta_I$ and $M \notin \bigcup_{i=1}^{n} \Delta_{I_i}$. Hence $M \supset I$ and M does not contain every $I_i (i = 1,2,\ldots,n)$. Therefore, since M is a maximal ideal, there are elements $a_i \in I_i$ and $m_i \in M$ such that

$$e = a_i + m_i (i = 1,2,\ldots,n).$$

Thus, we have

$$e = a_1 + a_2 + \ldots + a_n + m, m \in M$$

and $a_1 + a_2 + \ldots + a_n \in I$. This implies $I + M = R$. Hence, by $I \subset M$, we have $M = R$, which is a contradiction. This shows the following relation:

$$\bigcup_{i=1}^{n} \Delta_{I_i} = \Delta_I$$

and we have the following:

Theorem 3.3 *The closed sets for $\mathcal{M}_\Omega$ are expressed by sets Δ_I, where I is an ideal of R.*

By Theorem 3.3, we prove the following

Theorem 3.4 *The space $\mathcal{M}_\Omega$ is a compact T_1-space.*

Proof. Let $\{\Delta_{I_\lambda}\}$ be a family of closed sets in $\mathcal{M}_\Omega$ with the finite intersection property, where I_λ are ideals in R. Therefore, any finite family of I_λ does not contain the Γ-semiring R. Hence the ideal I generated by $\{I_\lambda\}$ does not contain the identity e of R. This shows that I is contained in a maximal ideal M. Hence $M \in \bigcap_\lambda \Delta_{I_\lambda}$. Therefore, since $\bigcap_\lambda \Delta_{I_\lambda}$ is non-empty, $\mathcal{M}_\Omega$ is a compact space.

On topological space of prime ideals of Γ-semiring

In this section, the structure space $\mathcal{B}$ of all prime ideals of a Γ-semiring R with identity e is considered and the relation of $\mathcal{B}$ and the structure space $\mathcal{M}$ of all maximal ideals of R is investigated. Throughout the section, we shall treat a commutative Γ-semiring R with identity e. An ideal P of R is prime if and only if $a\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$. Since R has an identity e, any maximal ideal is prime, therefore $\mathcal{B} \supseteq \mathcal{M}$.

To introduce a topology τ on $\mathcal{B}$, we shall take $\tau_x = \{P \mid x \notin P, P \in \mathcal{B}\}$ for every $x \in R$ as an open base of $\mathcal{B}$. We have the following

Theorem 4.1 *Let* $\mathcal{U}$ *be a subset of* $\mathcal{B}$*, then*

$$\overline{\mathcal{U}} = \{P' \in \mathcal{B} \mid \bigcap_{P \in \mathcal{U}} P \subset P'\},$$

where $\overline{\mathcal{U}}$ *is the closure of* $\mathcal{U}$ *by the topology* τ.

Proof. Let $P' \in \{P' \in \mathcal{B} \mid \bigcap_{P \in \mathcal{U}} P \subset P'\}$ and let τ_x be a neighbourhood of P', then $x \notin P'$, and we have $x \notin \bigcap_{P \in \mathcal{U}} P$. Therefore, there is a prime ideal $P \in \mathcal{U}$ such that P does not contain x and $\tau_x \ni P$. This shows that $P \in \overline{\mathcal{U}}$. Thus we have proved that the $\overline{\mathcal{U}}$ contains $\{P' \in \mathcal{B} \mid \bigcap_{P \in \mathcal{U}} P \subset P'\}$.

If a prime ideal P' is not in $\{P' \in \mathcal{B} \mid \bigcap_{P \in \mathcal{U}} P \subset P'\}$, then $\bigcap_{P \in \mathcal{U}} P - P' \neq \varnothing$. Hence, for $x \in \bigcap_{P \in \mathcal{U}} P - P'$, we have $x \in P, P \in \mathcal{U}$ and $x \notin P'$. This shows $P \notin \tau_x, P \in \mathcal{U}$ and $P' \notin \tau_x$. Therefore $\tau_x \cap \mathcal{U} = \varnothing$ and hence $P' \notin \overline{\mathcal{U}}$. The proof is complete.

A similar argument for $\mathcal{M}$ relative to Ω-topology implies the following

Proposition 4.2 *Let* $\mathcal{U}$ *be a subset of* $\mathcal{M}$ *, then*

$$\overline{\mathcal{U}} = \{M' \in \mathcal{M} \mid \bigcap_{M \in \mathcal{U}} M \subset M'\},$$

where $\overline{\mathcal{U}}$ *is the closure of* $\mathcal{U}$ *by the topology* Ω.

In a similar way to the proof of the Theorem 2.1, we can prove the following

Theorem 4.3 *The closure operation* $\mathcal{U} \to \overline{\mathcal{U}}$ *of* $\mathcal{B}$ *satisfies the following relations:*

1. $\mathcal{U} \subseteq \overline{\mathcal{U}}$.
2. $\overline{\overline{\mathcal{U}}} = \overline{\mathcal{U}}$.
3. $\overline{\mathcal{U} \cup \mathcal{B}} = \overline{\mathcal{U}} \cup \overline{\mathcal{B}}$.

Proof. We shall prove only the last relation (3). By Theorem 4.1, $\mathcal{U} \subset \mathcal{B}$ implies $\overline{\mathcal{U}} \subset \overline{\mathcal{B}}$ and hence $\overline{\mathcal{U}} \cup \overline{\mathcal{B}} \subset \overline{\mathcal{U} \cup \mathcal{B}}$. Let $P \notin \overline{\mathcal{U}} \cup \overline{\mathcal{B}}$, then $P \notin \overline{\mathcal{U}}$ and $P \notin \overline{\mathcal{B}}$. Hence $P \not\supset \bigcap_{P' \in \mathcal{U}} P' = P_{\mathcal{U}}$ and $P \not\supset \bigcap_{P' \in \mathcal{B}} P' = P_{\mathcal{B}}$. The sets $\mathcal{B}_{\mathcal{U}}$ and $\mathcal{B}_{\mathcal{B}}$ are ideals. If $P_{\mathcal{U}} \Gamma P_{\mathcal{B}} \subset P$, for any elements a, b such that $a \in P_{\mathcal{U}} - P, b \in P_{\mathcal{B}} - P$, we have $a\Gamma b \subseteq P$ and since P is a prime ideal, $a \in P$ or $b \in P$, which is a contradiction. Therefore, $P \not\supset P_{\mathcal{U}} \Gamma P_{\mathcal{B}} \supseteq P_{\mathcal{U}} \cap P_{\mathcal{B}} = P_{\mathcal{U} \cup \mathcal{B}}$. Hence $P \notin \overline{\mathcal{U} \cup \mathcal{B}}$.

Theorem 4.4 *The topological space* $\mathcal{B}$ *is a* T_0*-space.*

Proof. It is sufficient to prove that $(\overline{P_1}) = (\overline{P_2})$ implies $P_1 = P_2$. By $P_2 \in (\overline{P_1})$, then $P_2 \supset P_1$. Similarly $P_1 \supset P_2$ and we have $P_1 = P_2$.

Theorem 4.5 *The topological space* $\mathcal{B}$ *is a compact* T_1*-space.*

Proof. Let $\mathcal{U}_\lambda$ be a family of closed sets such that $\bigcap_\lambda \mathcal{U}_\lambda = \varnothing$, then we have $\sum P_{\mathcal{U}_\lambda} = R$, where $P_{\mathcal{U}_\lambda} = \bigcap_{P \in \mathcal{U}_\lambda} P$. Let us suppose that $\sum P_{\mathcal{U}_\lambda} \neq R$. Then there is a maximal ideal M containing $\sum P_{\mathcal{U}_\lambda}$. Therefore $\sum P_{\mathcal{U}_\lambda} \subset M$ for every λ. Hence $\mathcal{U}_\lambda \ni M$ for every λ, and we have $\bigcap_\lambda \mathcal{U}_\lambda \ni M$, which is a contradiction. By $\sum P_{\mathcal{U}_\lambda} = R$, we have there exit $\gamma_1, \gamma_2, ..., \gamma_n \in \Gamma$, such that $e = a_1\gamma_1 a_2\gamma_2 ... \gamma_n a_n, a_i \in P_{\mathcal{U}_{\lambda_i}}$ $(i = 1,2,...,n)$. Hence $\sum_{i=1}^{n} P_{\mathcal{U}_{\lambda_i}} = R$. If $\bigcap_{i=1}^{n} \mathcal{U}_{\lambda_i} \neq \varnothing$, then for a prime ideal P of $\bigcap_{i=1}^{n} \mathcal{U}_{\lambda_i}$, we have $P \supset P_{\mathcal{U}_{\lambda_i}}$ $(i = 1,2,...,n)$ and hence $P \supset \sum_{i=1}^{n} P_{\mathcal{U}_{\lambda_i}}$. Therefore we have $\bigcap_{i=1}^{n} \mathcal{U}_{\lambda_i} = \varnothing$.

By the $\mathcal{B}$*-radical* $r(\mathcal{B})$ of the Γ-semiring R, we mean the intersection of all prime ideals of R, that is, $\bigcap_{P \in \mathcal{B}} P$. By the $\mathcal{M}$ *-radical* $r(\mathcal{M})$ of R, we mean the intersection of all maximal ideals of R, that is, $\bigcap_{M \in \mathcal{M}} M$.

From $\mathcal{M} \subseteq \mathcal{B}$, we have $r(\mathcal{B}) \subseteq r(\mathcal{M})$. In the following proposition we give a condition to be $r(\mathcal{B}) = r(\mathcal{M})$.

Theorem 4.6 *The subset* $\mathcal{M}$ *of* $\mathcal{B}$ *is dense in* $\mathcal{B}$*, if and only if,* $r(\mathcal{B}) = r(\mathcal{M})$.

Proof. Let $\overline{\mathcal{M}} = \mathcal{B}$ for the topology τ. Then we have

$$\{P \mid \bigcap_{M \in \mathcal{M}} M \subset P\} = \mathcal{B}.$$

Hence

$$r(\mathcal{M}) = \bigcap_{M \in \mathcal{M}} M \subseteq \bigcap_{P \in \mathcal{B}} P = r(\mathcal{B}).$$

Since $r(\mathcal{B}) \subseteq r(\mathcal{M})$, therefore we have $r(\mathcal{B}) = r(\mathcal{M})$.

Conversely, if $P \in \mathcal{B} - \overline{\mathcal{M}}$, then $P \in \mathcal{B}$ and $P \in \overline{\mathcal{M}}$. Therefore, there is a neighbourhood τ_x of P such that $\tau_x \cap \mathcal{M} = \varnothing$. Hence $r(\mathcal{B}) = \bigcap_{P \in \mathcal{B}} P$ is a proper subset of $\bigcap_{M \in \mathcal{M}} M$. Therefore $r(\mathcal{B}) \neq r(\mathcal{M})$, which completes the proof.

Definition 4.7 *If* $r(\mathcal{M})$ *is the zero ideal (0), then* A *is said to be* $\mathcal{M}$ $-$*semisimple*.

From the Theorem 4.6, we have the following

Theorem 4.8 *If* R *is* $\mathcal{M}$ *-semisimple,* $\mathcal{M}$ *is dense in* $\mathcal{B}$.

On topological space of strongly irreducible ideals of Γ-semiring

In this section, the structure space $\mathcal{S}$ of all strongly irreducible ideals of a commutative Γ-semiring R with identity e is investigated.

An ideal I of a Γ-semiring R is called *irreducible*, if and only if $A \cap B = I$ for two ideals A, B implies $A = I$ or $B = I$. An ideal I of a Γ-semiring R is called *strongly irreducible*, if and only if $A \cap B \subset I$ for any two ideals A, B implies $A \subset I$ or $B \subset I$. From $A\Gamma B \subset A \cap B$ for any two ideals A, B, it follows that any prime ideals are strongly irreducible and any strongly irreducible ideals are irreducible.

Let S be the set of all strongly irreducible ideals of R. From the above, it is clear that $\mathsf{M} \subset \mathsf{B} \subset \mathsf{S}$. To give a topology σ on S, we shall take $\sigma_x = \{S \in \mathsf{S} \mid x \notin S\}$ for every $x \in R$ as an open base of S.

Theorem 5.1 *Let* U *be a subset of* S*, then we have*

$$\overline{\mathsf{U}} = \{S' \in \mathsf{S} \mid \bigcap_{S\in\mathsf{U}} S \subset S'\}$$

where $\overline{\mathsf{U}}$ *is the closure of* U *by* σ.

Proof. Let $\mathsf{F} = \{S' \in \mathsf{S} \mid \bigcap_{S\in\mathsf{U}} S \subset S'\}$ and let $S' \in \mathsf{F}$. Let σ_x be an open base of S', then, by the definition of the topology σ, $x \notin S'$. Hence, we have $x \notin \bigcap_{S\in\mathsf{U}} S$. It follows from this that there is a strongly irreducible ideal S of U such that x is not contained in S. Hence $\sigma_x \ni S$. Therefore $S' \in \overline{\mathsf{U}}$ and $\mathsf{F} \subset \overline{\mathsf{U}}$.

To prove that $\mathsf{F} \supset \overline{\mathsf{U}}$, take a strongly irreducible ideal S' such that $S' \notin \mathsf{F}$. Then $\bigcap_{S\in\mathsf{U}} S - S' \neq \varnothing$. For an element $x \in \bigcap_{S\in\mathsf{U}} S - S'$, we have $x \in S (S \in \mathsf{U})$ and $x \in S'$. Hence $S' \in \sigma_x$ and $S \notin \sigma_x$ for all S of U. Therefore $\mathsf{U} \cap \sigma_x = \varnothing$ and then we have $S' \notin \overline{\mathsf{U}}$. Hence $\mathsf{F} \supset \overline{\mathsf{U}}$. The proof of the theorem is complete.

We shall prove that the topological space S for the topology σ is a compact T_0-space. To prove that S is a T_0-space, it is sufficient to verify the following conditions:

1. $\mathsf{U} \subseteq \overline{\mathsf{U}}$.
2. $\overline{\overline{\mathsf{U}}} = \overline{\mathsf{U}}$.
3. $\overline{\mathsf{U} \cup \mathsf{B}} = \overline{\mathsf{U}} \cup \overline{\mathsf{B}}$
4. $\overline{S_1} = \overline{S_2}$ implies $S_1 = S_2$.

The conditions (1) and (2) are clear, and $\mathsf{U} \cup \mathsf{F}$ implies $\overline{\mathsf{U}} \subset \overline{\mathsf{F}}$. From this relation, we have $\overline{\mathsf{U}} \cup \overline{\mathsf{F}} \subset \overline{\mathsf{U} \cup \mathsf{F}}$. For some element of S of $\overline{\mathsf{U} \cup \mathsf{F}}$, suppose that $S \notin \overline{\mathsf{U}}$ and $\overline{S} \notin \mathsf{F}$. From Theorem 5.1, we have

$$S \not\supset \bigcap_{S'\in\mathsf{U}} S' = S_{\mathsf{U}} \text{ and } S \not\supset \bigcap_{S'\in\mathsf{F}} S' = S_{\mathsf{F}} .$$

S_{U} and S_{F} are ideals. If $S_{\mathsf{U}} \cap S_{\mathsf{F}} \subset S$, by the definition of S, $S_{\mathsf{U}} \subset S$ or $S_{\mathsf{F}} \subset S$. Hence $S \not\supset S_{\mathsf{U}} \cap S_{\mathsf{F}} = S_{\mathsf{U}\cup\mathsf{F}}$. This showa $S \notin \overline{\mathsf{U} \cup \mathsf{F}}$.

To prove that $\overline{S_1} = \overline{S_2}$ implies $S_1 = S_2$, we shall use the condition (1). Then $\overline{S_1} \ni S_2$ and by the definition of closure operation, we have $S_1 \subset S_2$. Similarly we have $S_1 \supset S_2$ and $S_1 = S_2$. Therefore we complete the proof that S is a T_0-space.

We shall prove that S is a compact space. Let U_t be a family of closed sets with empty intersection. Let $S_{\mathsf{U}_t} = \bigcap_{S\in\mathsf{U}_t} S$, suppose that $\sum_t S_{\mathsf{U}_t} \neq S$, then there is a maximal ideal M containing the ideal $\sum_t S_{\mathsf{U}_t}$. Therefore we have $S_{\mathsf{U}_t} \subset M$ for every t. By the definition of $S_{\mathsf{U}_t}, \mathsf{U}_t \ni M$ for every t. Hence $\bigcap_t \mathsf{U}_t \ni M$, which contradicts our hypothesis of U_t. Therefore $\sum_t S_{\mathsf{U}_t} = R$. Hence we have there exist $\gamma_1, \gamma_2, ..., \gamma_n \in \Gamma$, such that $e = a_1\gamma_1 a_2\gamma_2 ... \gamma_n a_n (a_i \in S_{\mathsf{U}_{t_i}} \ (i = 1,2,...,n))$. Hence we have $R=\sum_{i=1}^{n} S_{U_{t_i}}$. If $\bigcap_{i=1}^{n} U_{t_i} \neq \varnothing$, for every strongly irerducible ideal S of $\bigcap_{i=1}^{n} U_{t_i}$, $S \supset S_{U_{t_i}}$ $(i=1,2,...,n)$ and $S \supset \sum_{i=1}^{n} S_{U_{t_i}}$. If $\bigcap_{i=1}^{n} U_{t_i} = R$, we can prove easily that $\mathcal{S}$ is a compact space. If $\bigcap_{i=1}^{n} U_{t_i}$ contains a proper strongly irreducible ideal S, we have $S \supset \sum_{i=1}^{n} S_{U_{t_i}}$, which is a contradiction to $R=\sum_{i=1}^{n} S_{U_{t_i}}$. Therefore $\bigcap_{i=1}^{n} U_{U_{t_i}} = \varnothing$. Hence $\mathcal{S}$ is a compact space. Thus we have proved the following

Theorem 5.2 *The topological space* $(\mathcal{S},\sigma)$ *is compact* T_0*-space.*

By the $\mathcal{S}-radical$ $r(\mathcal{S})$ of a Γ-semiring, we mean the intersection of all strongly irreducible ideals of it, i.e., $\bigcap_{S\in\mathcal{S}} S$. from $\mathcal{M} \subset \mathcal{B} \subset \mathcal{S}$, we have $r(\mathcal{M}) \supset r(\mathcal{B}) \supset r(\mathcal{S})$.

Theorem 5.3 *The subset* $\mathcal{B}$ *of* $\mathcal{S}$ *is dense in* $\mathcal{S}$*, if and only if* $r(\mathcal{B}) = r(\mathcal{S})$.

Proof. Let $\overline{\mathcal{B}} = \mathcal{S}$ for the topology σ, then we have

$$\{S \mid \bigcap_{P\in\mathcal{B}} P \subset S\} = \mathcal{S}.$$

Hence, we have

$$r(\mathcal{B}) = \bigcap_{P\in\mathcal{B}} P \subset \bigcap_{S\in\mathcal{S}} S = r(\mathcal{S}).$$

On the other hand, $r(\mathcal{B}) \supset r(\mathcal{S})$. This shows $r(\mathcal{S}) = r(\mathcal{B})$.

Conversely, suppose that $\mathcal{S} - \overline{\mathcal{B}} \neq \varnothing$, then there is a strongly irreducible ideal S such that $S \notin \overline{\mathcal{B}}$ and $S \in \mathcal{S}$. Therefore there is a neighbourhood σ_x of S which does not meet $\mathcal{B}$. Hence $r(\mathcal{S}) = \bigcap_{S\in\mathcal{S}} S$ is a proper subset of $\bigcap_{P\in\mathcal{B}} P$, and we have $r(\mathcal{S}) \neq r(\mathcal{B})$.

Corollary 5.4 *The subset* $\mathcal{M}$ *of* $\mathcal{S}$ *is dense in* $\mathcal{S}$*, if and only if* $r(\mathcal{M}) = r(\mathcal{S})$.

Corollary 5.5 *Let* R *be a* Γ*-semiring with 0. If* R *is* $\mathcal{M}$ *-semisimple, then* $\mathcal{M}$ *and* $\mathcal{B}$ *are dense in* $\mathcal{S}$.

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