COMMON FIXED POINT THEOREMS FOR COMMUTING MAPPINGS ON A QUASIMETRIC SPACE

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Abstract:
A general fixed point theorem for commuting mappings on quasimetric spaces is proved. Let (X,d) be a complete quasimetric space with a constant $\beta \geq 1$ and let f,g be commuting mappings form *X* into itself. If *f* is continuous and satisfies the following condition:

$$g(X) \subset f(X).$$

and further, there exist $\alpha \in (0, \frac{1}{\beta})$ such that for all $x, y \in X$,

$$d(gx, gy) \leq \alpha \max\{d(fx, fy), d(fx, gx), d(fx, gy), d(fy, gx), d(fy, gy)\}$$

then *f* and *g* have a unique common fixed point in *X*.
This result generalizes and extends the main theorem from [1] and [2]. Some new results concerning fixed points for commuting mappings on quasimetric spaces are obtained too which extend the results obtained in [3].

Key Words: Fixed point, quasimetric space, complete quasimetric space, common fixed point

Introduction and preliminaries

In [1], Jungck proved the following fixed point theorem:

Theorem 1.1 *Let (X,d) be a complete metric space. Let f and g be commuting continuous self-mappings on X such that*

$$g(X) \subset f(X).$$

Further, let there exist a constant $a \in (0,1)$ such that for all $x, y \in X$:

$$d(gx, gy) \leq a d(fx, fy).$$

Then f and g have a unique common fixed point in X .

In [2], C'iric' proved the following fixed point theorem:

Theorem 1.2(C'iric') *Let (X,d) be a complete metric space. Let f be a self-mapping on X such that for some constant $a \in (0,1)$ and for every $x, y \in X$*

$$d(fx, fy) \leq a \max\{d(x,y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}.$$

Then f possesses a unique fixed point in X .

In [3] some results concerning fixed points on a metric space are obtained, which generalize and unify fixed point theorems in [1] and [2].

In this paper we obtain some new results concerning fixed points for mappings on quasimetric spaces. We generalize and extend the results of the Theorem 1.1 and 1.2 for commuting mappings on quasimetric spaces. We also extend the results established in [3] for commuting mappings on quasimetric spaces.

First, we give a standard definition and notation which will be used in the sequel.

Definition 1.3 [4] *Let X be an arbitrary set and R^+ the set of nonnegative real numbers. A function $d : X \times X \to R^+$ is called a quasidistance on X if and only if there exists a constant $k \geq 1$, such that for all x, y and $z \in X$ the following conditions hold:*

(1) $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq k[d(x, z) + d(z, y)]$.

Inequality (3) is often called *quasitriangular inequality* and k is often called the *quasitriangular constant* of d. Of course, d is called a metric when $k = 1$.

A pair (X, d) is called *quasimetric space* if X is a set and d is a quasidistance on X. It is clear that for $k = 1$ we obtain the metric space.

The following example illustrate the existence of the quasidistance.

Example 1.4 Let $X = R \times R$ and $x = (x_1, x_2) \in X$, $y = (y_1, y_2) \in X$. The function $d : X \times X \to R^+$ such that

$$d(x, y) = \begin{cases} k|x_1 - y_1| + |x_2 - y_2|, \ for |x_1 - y_1| \leq |x_2 - y_2| \\ |x_1 - y_1| + k|x_2 - y_2|, \ for |x_1 - y_1| > |x_2 - y_2| \end{cases}$$

is a quasidistance with $k \geq 1$.

Let we verify the satisfying of the tree conditions of the definiton 1.3.

1.

$$d(x, y) = 0 \Leftrightarrow \begin{cases} x_1 - y_1 = 0 \\ x_2 - y_2 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_2 \end{cases} \Leftrightarrow x = (x_1, x_2) = y = (y_1, y_2)$$

2. $d(x, y) = d(y, x), \forall x, y \in X$ since $|x_1 - y_1| = |y_1 - x_1|$ and $|x_2 - y_2| = |y_2 - x_2|$.

3.

$$\begin{aligned} d(x, y) &\leq k\left[|x_1 - y_1| + |x_2 - y_2|\right] = k\left[|x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2|\right] \leq \\ &\leq k\left[|x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2|\right] = \\ &= k\left[|x_1 - z_1| + |x_2 - z_2|\right] + k\left[|z_1 - y_1| + |z_2 - y_2|\right] \leq \\ &\leq k\left[d(x, z) + d(z, y)\right], \forall x, y, z \in X. \end{aligned}$$

Main results

Let (X, d) be a complete quasimetric space with a constant $\beta \geq 1$ and let f, g be commuting mappings from X into itself. Let f be continuous and satisfies the following condition:

$$g(X) \subset f(X) \qquad (1)$$

and further, there exist $\alpha \in (0, \frac{1}{\beta})$ such that for all $x, y \in X$,

$$d(gx, gy) \leq \alpha \ \max\{d(fx, fy), d(fx, gx), d(fx, gy), d(fy, gx), d(fy, gy)\} \qquad (2).$$

Let x_0 be an arbitrary point in X and define the sequence (x_n) in X as follows: By the condition $g(X) \subset f(X)$ it follows that there exists $x_1 \in X$ such that $gx_0 = fx_1$. In the same way we define succesively $x_2, ..., x_n$. Let $x_{n+1} \in X$ such that

$$gx_n = fx_{n+1} = y_n; n = 1, 2, ...$$

We denote

$$G(y_k; n) = \{y_k, y_{k+1}, ..., y_{k+n}\}.$$

We denote by $\delta(G(y_k; n))$ the diameter of the set $G(y_k; n)$.

We shall prove the following lemma which is necessary to prove our main theorem.

Lemma 2.1 *If* $\delta(G(y_k;n)) > 0, k \geq 0, n \in N$ *, then*

(1) $\delta(G(y_k;n)) = d(y_k, y_j)$ with $k < j \leq k+n$.

(2) $\delta(G(y_k;n)) \leq \alpha\delta(G(y_{k-1};n+1))$.

(3) $\delta(G(y_0;n+k)) \leq \dfrac{\beta}{1-\beta\alpha} d(y_0, y_1)$.

Proof. For i and j such that $1 \leq i < j$, we have

$$d(y_i, y_j) = d(gx_i, gx_j) \leq$$
$$\leq \alpha max\{d(fx_i, fx_j), d(fx_i, gx_j), d(x_j, gx_j, d(fx_i, gx_j), d(fx_j, gx_i)\}$$
$$= \alpha max\{d(y_{i-1}, y_{j-1}), d(y_{i-1}, y_i), d(y_{i-1}, y_j, d(y_{i-1}, y_j), d(y_{j-1}, y_i)\}$$

Thus

$$d(y_i, y_j) \leq \alpha\delta(G(y_{i-1}, j-i+1)) \qquad (3)$$

By using the notion of the superior of a finite number distances we have

$$\delta(G(y_k;n)) = d(y_i, y_j)$$

for some pair i, j such that $k \leq i < j \leq k+n$.

If $i > k$, by (3) we have

$$\delta(G(y_k, n)) \leq \alpha\delta(G(y_{i-1}, j-i+1))$$

with $i-1 \geq k$ and $j \leq k+n$. Then

$$\delta(G(y_k, n)) \leq \alpha\delta(G(y_k, n))$$

a contradiction, since $0 \leq \alpha < \dfrac{1}{\beta} \leq 1$.

Thus we have $i = k$ and this completes the proof of (a). Further, during the proof of lemma is proved that

$$\delta(G(y_k, n)) = d(y_k, y_j) \leq \alpha\delta(G(y_{k-1}, j-k+1)) \leq \alpha\delta(G(y_{k-1}, n+1)).$$

Thus, (b) holds too. Now, let we prove (c).

By (a) and using quasitriangular inequality on quasimetric spaces we have

$$d(G(y_l;m)) = d(y_l, y_j) \leq \beta[d(y_l, y_{l+1}) + d(y_{l+1}, y_j)] =$$
$$= \beta d(y_l, y_{l+1}) + \beta d(y_{l+1}, y_j) \leq \beta d(y_l, y_{l+1}) + \beta\delta(G(y_{l+1}, m-1))$$

with $l < j \leq l+m$.

Using (b) we have

$$\delta(G(y_l;m)) \leq \beta d(y_l, y_{l+1}) + \beta\alpha\delta G(y_l;m)$$

which implies that

$$\delta(G(y_l;m)) \leq \frac{\beta}{1-\beta\alpha} d(y_l, y_{l+1}) \qquad (4)$$

since $1-\beta\alpha > 0$.

Using again (b) we have

$$\delta(G(y_k, n)) \leq \alpha\delta(G(y_{k-1}, n+1)) \leq \alpha^2\delta(G(y_{k-2}, n+2)) \leq ... \leq \alpha^k\delta(G(y_0;n+k))$$

For $l = 0$ and $m = n+k$, by (4) we have

$$\delta(G(y_0;n+k)) \leq \frac{\beta}{1-\beta\alpha} d(y_0, y_1).$$

Finally,

$$\delta(G(y_k;n)) \le \frac{\alpha^k \beta}{1-\beta\alpha} d(y_0, y_1).$$

This completes the proof of Lemma.

Now we give and prove our theorem as follows

Theorem 2.2 *Let (X,d) be a complete quasimetric space with a constant $\beta \ge 1$, quasidistance d continuous and let f,g be commuting mappings form X into itself. If f is continuous and satisfies the following condition:*

$$g(X) \subset f(X).$$

and further, there exist $\alpha \in (0, \frac{1}{\beta})$ such that for all $x, y \in X$, $d(gx, gy) \le \alpha$ max $\{d(fx, fy), d(fx, gx), d(fx, gy), d(fy, gx), d(fy, gy)\}$. Then f and g have a unique common fixed point in X.

Proof. It is enough to find a point y such that

$$fy = gy.$$

Applying (2) for $x = gy$ we have

$$d(ggy, gy) \le \alpha max\{d(fgy, fy), d(fgy, ggy), d(fy, gy), d(fgy, gy), d(fy, ggy)\} = \alpha d(ggy, gy).$$

By the inequality $d(ggy, gy) \le \alpha d(ggy, gy)$, since $0 < \alpha < \frac{1}{\beta} \le 1$ it follows

$$d(ggy, gy) = 0.$$

This shows that $ggy = gy$. Thus, gy is a fixed point of g. On the other hand, since f and g commute we have

$$fgy = gfy = ggy = gy$$

which implies that gy is also a fixed point of f.

Case 1. Let $\delta(G(y_k;n)) = 0$ for some n and k. Then we have $y_k = y_{k+1}$ or

$$fx_{k+1} = gx_{k+1}.$$

It is clear that y_{k+1} is a common fixed point of f and g.

Case 2. Let $\delta(G(y_k;n)) > 0$.

For $n < m$ we consider the distance

$$d(y_m, y_n) \le \delta(G(y_n, m-n)) \le \alpha^n \frac{\beta}{1-\beta\alpha} d(y_0, y_1)$$

and $\lim_{n\to\infty} d(y_m, y_n) = 0$. Thus, (y_n) is a Cauchy sequence with a limit y in X, since (X,d) is a complete quasimetric space.

Since f is continuous it follows that

$$\lim_{n\to\infty} fy_n = fy$$

and using the equality $gy_n = fy_{n+1}$ we have

$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} fy_{n+1} = fy.$$

On the other hand we have

$$d(fy_{n+1}, gy) = d(gy_n, gy) \le$$
$$\le \alpha max\{d(fy_n, fy), d(fy_n, gy_n), d(fy, gy), d(fy_n, gy), d(fy_n, gy_n)\}.$$

Letting n tend to infinity, we get

$$g(fy, gy) \le \alpha d(fy, gy)$$

which holds only for $fy = gy$, since $\alpha < 1$. By (2) it follows that y is a unique common fixed point of f and g.

Corollary 2.3 *The Theorem 1.2 (C′iric′) is a corollary of the Theorem 2.2. If $f = I_x$, where I_x is the identity mapping in X, then the condition (2) just as in Lemma 2.1 can be written in the following form:*

$$d(gx, gy) \le \alpha \max\{d(x, y), d(x, gx), d(y, gy), d(x, gy), d(y, gx)\}$$

If we replace g by f we obtain the condition of the Theorem 1.2(C′iric′)

Corollary 2.4 *The Theorem 1.1 is a corollary of the Theorem 2.2.*

It is obvious that whenever the condition $d(gx, gy) \le \alpha d(fx, fy)$ is satisfied, the condition (2) is satisfied and so the corollary follows.

Corollary 2.5 *The Theorem 2.2 is a generalization of the Banach Theorem.*

If we take $k = 1$ and $f = I_x$ then by Corollary 2.4 the corollary follows.

Corollary 2.6 *The Theorems 1.1 and 1.2 can be proved on the quasimetric spaces with the constant β, where $0 < \alpha < \frac{1}{\beta}$.*

In the following theorem, as the domain of g is considered $f(X)$. We also replace the continuously condition of f by the continuously of $f^2 = ff$ as an weakly condition.

Theorem 2.7 *Let (X, d) be a complete quasimetric space with the constant $\beta \ge 1$, quasidistance d continuous, let f be a mapping from X into itself such that f^2 is continuous. Let*

$$g : f(X) \to X$$

be a mapping such that

$$gf(X) \subset f^2(X) \qquad (5)$$

and for all x from the domain of fg and gf, $fgx = gfx$. Further, there exist $\alpha \in (0, \frac{1}{\beta})$ such that for all $x, y \in f(X)$ the condition (2) is satisfied.

Then, f and g have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in $f(X)$. Let x_1 be a point in $f(X)$ such that

$$gx_0 = fx_1 = y_0.$$

This follows by (5). In the same way we define succesively $x_2, ..., x_n$. Let $x_{n+1} \in f(X)$ such that

$$gx_n = fx_{n+1} = y_n; n = 1, 2, ...$$

We denote:

$$fy_n = fgx_n = ffx_{n+1} = gy_{n-1} = z_n.$$

For $k > 0$ and $n \in N$, one can prove, just as in Lemma 2.1, that

$$\delta(G(z_n; n)) \le \frac{\alpha^k \beta}{1 - \alpha\beta} d(z_0, z_1).$$

Therefore, (z_n) is a Cauchy sequence with a limit z in X.

Since f^2 is continuous it follows that

$$\lim_{n\to\infty} f^2 z_n = f^2 z$$

Further, we have

$$gfz_n = gff^2 x_{n+1} = f^2 fgx_{n+1} = f^2 z_{n+1}$$

which implies that

$$\lim_{n\to\infty} gfz_n = f^2 z.$$

Now by the condition (2) of the Theorem 2.2 we have

$$d(f^2z_{n+1}, gfz) = d(gfz_n, gfz) \le$$
$$\le \alpha max\{d(f^2z_n, f^2z), d(f^2z_n, gfz_n), d(f^2z, gfz), d(f^2z_n, gfz), d(f^2z, gfz_n)\}.$$

Letting n tend to infinity, we get

$$d(f^2z, gfz) \le \alpha d(f^2z, gfz)$$

which holds only in the case when $f^2z = gfz$.

Finally, by

$$d(ggfz, gfz) \le \alpha max\{d(fgfz, f^2z), d(fgfz, ggfz), d(f^2z, gfz), d(fgfz, gfz), d(f^2z, ggfz)\}$$
$$= \alpha d(gfz, gfz)$$

it follows that $ggfz = gfz$. Therefore, gfz is a fixed point of g.

On the other hand, since we have

$$fgfz = f^2gz = gfgz = ggfz = gfz$$

then gfz is also a fixed point of f. Therefore f and g have a common fixed point. By the condition (2) it follows that it is the unique common fixed point of f and g.

Theorem 2.8 *Let (X, d) be a complete quasimetric space with the constant $\beta \ge 1$, quasidistance d continuous. If f, g are commuting mappings form X into itself such that f^2 is continuous satisfying the following condition*

$$g(X) \subset f(X)$$

and further, there exist $\alpha \in (0, \frac{1}{\beta})$ such that for all $x, y \in X$

$$d(gx, gy) \le \alpha \ max\ \{d(fx, fy), d(fx, gx), d(fx, gy), d(fy, gx), d(fy, gy)\},$$

then f and g have a unique common fixed point in X.

Proof. By the condition $g(X) \subset f(X)$ follows

$$gf(X) = fg(X) \subset f^2(X)$$

which is the condition (5) of the Theorem 2.7. Since the condition (2) is satisfied, then by Theorem 2.2 and Theorem 2.7 it follows that f and g have a unique common fixed point.

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