PLANE STATE PROBLEM ANALYSIS WITH FINITE-DIFFERENCE METHOD

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Abstract:
This paper presents a finite-difference analysis of stresses and displacements of the plane elastic problems of orthotropic materials. Starting from the Airy stress function, we assume that in the case of orthotropic materials there is a function $\Psi(x, y)$ the partial derivatives of which determine the specific deformations with the material equations. We also use a potential function of the displacements, the partial derivatives of which lead to the stress fields with the help of the material equations. This will make the prescription of the mixed boundary conditions possible. Therefore, the description of the boundary conditions under the form of prescribed stresses (of the load distribution on the boundary) becomes possible, because there is a direct relation (differential equations) between the displacements and stresses.

Key Words: Displacement potential function, finite-difference method, orthotropic material

Introduction

In the linear elasticity theory it is assumed that the relations between stress and deformations are linear. This relation can be described by the formula

$$\{\boldsymbol{\sigma}\} = [\boldsymbol{E}] \cdot \{\boldsymbol{\varepsilon}\}, \tag{1}$$

where $[\boldsymbol{E}]$ is the elasticity matrix, a 6-by-6 matrix, which contains 36 material constants (Szalai, 1994).

For orthotropic materials in plane stress state, and in plain strain state, the two elasticity matrices $[\boldsymbol{E}]$ are valid only if the directions of orthotropy coincide with the directions of the coordinate axes. Otherwise, the two arrays must be rotated (Curtu, 1984). Thus, the transformation leads to a full matrix (Kakucs, 2007)

$$[\boldsymbol{E}] = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{E}_{13} \\ \bar{E}_{21} & \bar{E}_{22} & \bar{E}_{23} \\ \bar{E}_{31} & \bar{E}_{32} & \bar{E}_{33} \end{bmatrix}, \tag{2}$$

of which components are:

$$
\begin{aligned}
\overline{E}_{11} &= E_{11}\cdot\cos^4\theta + E_{22}\cdot\sin^4\theta + (E_{12}+E_{21}+4\cdot E_{33})\cdot\sin^2\theta\cdot\cos^2\theta,\\
\overline{E}_{12} &= E_{12}\cdot\cos^4\theta + E_{21}\cdot\sin^4\theta + (E_{11}+E_{22}-4\cdot E_{33})\cdot\sin^2\theta\cdot\cos^2\theta,\\
\overline{E}_{13} &= (E_{11}-E_{12}-2\cdot E_{33})\cdot\sin\theta\cdot\cos^3\theta + (E_{21}-E_{22}+2\cdot E_{33})\cdot\sin^3\theta\cdot\cos\theta,\\
\overline{E}_{21} &= E_{12}\cdot\sin^4\theta + E_{21}\cdot\cos^4\theta + (E_{11}+E_{22}-4\cdot E_{33})\cdot\sin^2\theta\cdot\cos^2\theta,\\
\overline{E}_{22} &= E_{11}\cdot\sin^4\theta + E_{22}\cdot\cos^4\theta + (E_{21}+E_{12}+4\cdot E_{33})\cdot\sin^2\theta\cdot\cos^2\theta,\\
\overline{E}_{23} &= (E_{11}-E_{12}-2\cdot E_{33})\cdot\sin^3\theta\cdot\cos\theta + (E_{21}-E_{22}+2\cdot E_{33})\cdot\sin\theta\cdot\cos^3\theta,\\
\overline{E}_{31} &= (E_{11}-E_{21}-2\cdot E_{33})\cdot\sin\theta\cdot\cos^3\theta + (E_{12}-E_{22}+2\cdot E_{33})\cdot\sin^3\theta\cdot\cos\theta,\\
\overline{E}_{32} &= (E_{11}-E_{21}-2\cdot E_{33})\cdot\sin^3\theta\cdot\cos\theta + (E_{12}-E_{22}+2\cdot E_{33})\cdot\sin\theta\cdot\cos^3\theta,\\
\overline{E}_{33} &= (E_{11}-E_{12}-E_{21}+E_{22}-2\cdot E_{33})\cdot\sin^2\theta\cdot\cos^2\theta + E_{33}\cdot(\sin^4\theta+\cos^4\theta),
\end{aligned}
\tag{3}
$$

where θ is the angle measured from the first direction of the orthotropy 1 to the x axis (figure 1.)

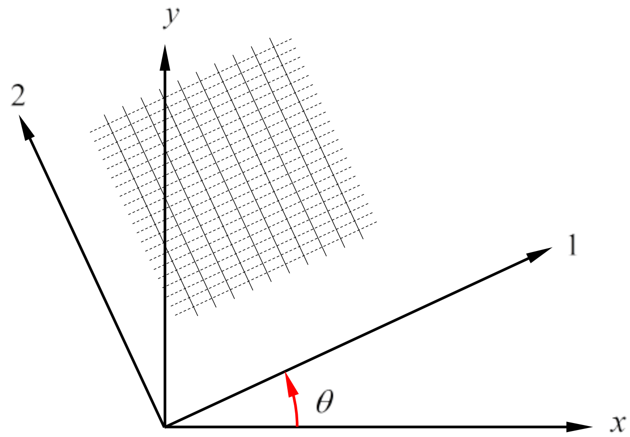

Fig. 1. Orthotropic direction

and

$$
[\boldsymbol{E}_{12}] = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & E_{33} \end{bmatrix}, \tag{4}
$$

is the matrix of elasticity in the directions of orthotropy (it has only three independent elements). Therefore, where the directions of orthotrophy do not coincide with the axes, the elasticity matrix contains nine nonzero elements and is symmetric. In the general case of plane anisotropy, the elasticity matrix is also a full and symmetric, but it contains six independent elements.

Formulation for finite-difference solution

Starting from the idea of Airy stress function, we suppose that in the case of orthotropic materials there is a function $\Psi(x,y)$ of which partial derivatives give the projections of the displacement. We transcribe the equilibrium equations using Hooke's law in specific strains (considering $f_x = 0$), then with the geometrical equations, we obtain the followings (Harangus, 2012):

$$\begin{aligned}&(\alpha_1\cdot E_{11}+\alpha_4\cdot E_{13})\cdot\frac{\partial^4\Psi}{\partial x^4}+\\&+(\alpha_1\cdot E_{13}+\alpha_1\cdot E_{31}+\alpha_2\cdot E_{11}+\alpha_4\cdot E_{12}+\alpha_4\cdot E_{33}+\alpha_5\cdot E_{13})\cdot\frac{\partial^4\Psi}{\partial x^3\cdot\partial y}+\\&+(\alpha_1\cdot E_{33}+\alpha_2\cdot E_{13}+\alpha_2\cdot E_{31}+\alpha_3\cdot E_{11}+\alpha_4\cdot E_{32}+\alpha_5\cdot E_{12}+\alpha_5\cdot E_{33}+\alpha_6\cdot E_{13})\cdot\frac{\partial^4\Psi}{\partial x^2\cdot\partial y^2}+\\&+(\alpha_2\cdot E_{33}+\alpha_3\cdot E_{13}+\alpha_3\cdot E_{31}+\alpha_5\cdot E_{32}+\alpha_6\cdot E_{12}+\alpha_6\cdot E_{33})\cdot\frac{\partial^4\Psi}{\partial x\cdot\partial y^3}+\\&+(\alpha_3\cdot E_{33}+\alpha_6\cdot E_{32})\cdot\frac{\partial^4\Psi}{\partial y^4}=0,\end{aligned}\tag{5}$$

$$\begin{aligned}&(\alpha_1\cdot E_{31}+\alpha_4\cdot E_{33})\cdot\frac{\partial^4\Psi}{\partial x^4}+\\&+(\alpha_1\cdot E_{21}+\alpha_1\cdot E_{33}+\alpha_2\cdot E_{31}+\alpha_4\cdot E_{23}+\alpha_4\cdot E_{32}+\alpha_5\cdot E_{33})\cdot\frac{\partial^4\Psi}{\partial x^3\cdot\partial y}+\\&+(\alpha_1\cdot E_{23}+\alpha_2\cdot E_{21}+\alpha_2\cdot E_{33}+\alpha_3\cdot E_{31}+\alpha_4\cdot E_{22}+\alpha_5\cdot E_{23}+\alpha_5\cdot E_{32}+\alpha_6\cdot E_{33})\cdot\frac{\partial^4\Psi}{\partial x^2\cdot\partial y^2}+\\&+(\alpha_2\cdot E_{23}+\alpha_3\cdot E_{21}+\alpha_3\cdot E_{33}+\alpha_5\cdot E_{22}+\alpha_6\cdot E_{23}+\alpha_6\cdot E_{32})\cdot\frac{\partial^4\Psi}{\partial x\cdot\partial y^3}+\\&+(\alpha_3\cdot E_{23}+\alpha_6\cdot E_{22})\cdot\frac{\partial^4\Psi}{\partial y^4}+f_y=0.\end{aligned}\tag{6}$$

We determine the α coefficients in such manner to get the multipliers of the partial derivatives of the first equation equal to zero (in this case any function Ψ is a solution of the first equation):

$$\begin{aligned}&\alpha_1\cdot E_{11}+\alpha_4\cdot E_{13}=0\\&\alpha_1\cdot E_{13}+\alpha_1\cdot E_{31}+\alpha_2\cdot E_{11}+\alpha_4\cdot E_{12}+\alpha_4\cdot E_{33}+\alpha_5\cdot E_{13}=0\\&\alpha_1\cdot E_{33}+\alpha_2\cdot E_{13}+\alpha_2\cdot E_{31}+\alpha_3\cdot E_{11}+\alpha_4\cdot E_{32}+\alpha_5\cdot E_{12}+\alpha_5\cdot E_{33}+\alpha_6\cdot E_{13}=0\\&\alpha_2\cdot E_{33}+\alpha_3\cdot E_{13}+\alpha_3\cdot E_{31}+\alpha_5\cdot E_{32}+\alpha_6\cdot E_{12}+\alpha_6\cdot E_{33}=0\\&\alpha_3\cdot E_{33}+\alpha_6\cdot E_{32}=0.\end{aligned}\tag{7}$$

Since six coefficients cannot be determined from these five equations, we must prescribe one of the values (Reaz, 2005). Therefore, we assign $\alpha_2=1$, and the five remaining coefficients are found by solving the system of equations (7), which can be solved by numerical methods. With the obtained α coefficients the second equilibrium equation will be the following:

$$\beta_1\cdot\frac{\partial^4\Psi}{\partial x^4}+\beta_2\cdot\frac{\partial^4\Psi}{\partial x^3\cdot\partial y}+\beta_3\cdot\frac{\partial^4\Psi}{\partial x^2\cdot\partial y^2}+\beta_4\cdot\frac{\partial^4\Psi}{\partial x\cdot\partial y^3}+\beta_5\cdot\frac{\partial^4\Psi}{\partial y^4}=\beta_0\cdot f_y\,,\tag{8}$$

of which solution is the potential function sought by us. The coefficients of this equation are:

$$\begin{aligned}&\beta_1=\alpha_1\cdot E_{31}+\alpha_4\cdot E_{33},\\&\beta_2=\alpha_1\cdot(E_{21}+E_{33})+\alpha_2\cdot E_{31}+\alpha_4\cdot(E_{23}+E_{32})+\alpha_5\cdot E_{33},\\&\beta_3=\alpha_1\cdot E_{23}+\alpha_2\cdot(E_{21}+E_{33})+\alpha_3\cdot E_{31}+\alpha_4\cdot E_{22}+\alpha_5\cdot(E_{23}+E_{32})+\alpha_6\cdot E_{33},\\&\beta_4=\alpha_2\cdot E_{23}+\alpha_3\cdot(E_{21}+E_{33})+\alpha_5\cdot E_{22}+\alpha_6\cdot(E_{23}+E_{32}),\\&\beta_5=\alpha_3\cdot E_{23}+\alpha_6\cdot E_{22},\\&\beta_0=-1.\end{aligned}\tag{9}$$

In case that the orthotropy directions coincide with axes x and y, the expression of coefficients β_i simplifies and solving the equation becomes easier. Thus, if the angle θ is an integer multiple of the right angle, the coefficients β_2 and β_4 are equal to zero.

The problem is ultimately reduced to solving the equation (8); we propose the solving with finite differences. Therefore, in the point of coordinates (x,y), which is the point (i,j) of the grid for calculating finite differences, we can write the following equation:

$$\begin{gathered}
\frac{\beta_4}{4\cdot h\cdot k^3}\cdot\Psi(i-1,j-2)+\frac{\beta_5}{k^4}\cdot\Psi(i,j-2)-\frac{\beta_4}{4\cdot h\cdot k^3}\cdot\Psi(i+1,j-2)+\\
+\frac{\beta_2}{4\cdot h^3\cdot k}\cdot\Psi(i-2,j-1)-\left(\frac{\beta_2}{2\cdot h^3\cdot k}-\frac{\beta_3}{h^2\cdot k^2}+\frac{\beta_4}{2\cdot h\cdot k^3}\right)\cdot\Psi(i-1,j-1)-\\
-\left(2\cdot\frac{\beta_3}{h^2\cdot k^2}+4\cdot\frac{\beta_5}{k^4}\right)\cdot\Psi(i,j-1)+\left(\frac{\beta_2}{2\cdot h^3\cdot k}+\frac{\beta_3}{h^2\cdot k^2}+\frac{\beta_4}{2\cdot h\cdot k^3}\right)\cdot\Psi(i+1,j-1)-\\
-\frac{\beta_2}{4\cdot h^3\cdot k}\cdot\Psi(i+2,j-1)+\frac{\beta_1}{h^4}\cdot\Psi(i-2,j)-\left(4\cdot\frac{\beta_1}{h^4}+2\cdot\frac{\beta_3}{h^2\cdot k^2}\right)\cdot\Psi(i-1,j)+\\
+\left(6\cdot\frac{\beta_1}{h^4}+4\cdot\frac{\beta_3}{h^2\cdot k^2}+6\cdot\frac{\beta_5}{k^4}\right)\cdot\Psi(i,j)-\left(4\cdot\frac{\beta_1}{h^4}+2\cdot\frac{\beta_3}{h^2\cdot k^2}\right)\cdot\Psi(i+1,j)+\\
+\frac{\beta_1}{h^4}\cdot\Psi(i+2,j)-\frac{\beta_2}{4\cdot h^3\cdot k}\cdot\Psi(i-2,j+1)+\\
+\left(\frac{\beta_2}{2\cdot h^3\cdot k}+\frac{\beta_3}{h^2\cdot k^2}+\frac{\beta_4}{2\cdot h\cdot k^3}\right)\cdot\Psi(i-1,j+1)-\left(2\cdot\frac{\beta_3}{h^2\cdot k^2}+4\cdot\frac{\beta_5}{k^4}\right)\cdot\Psi(i,j+1)-\\
-\left(\frac{\beta_2}{2\cdot h^3\cdot k}-\frac{\beta_3}{h^2\cdot k^2}+\frac{\beta_4}{2\cdot h\cdot k^3}\right)\cdot\Psi(i+1,j+1)+\frac{\beta_2}{4\cdot h^3\cdot k}\cdot\Psi(i+2,j+1)-\\
-\frac{\beta_4}{4\cdot h\cdot k^3}\cdot f(i-1,j+2)+\frac{\beta_5}{k^4}\cdot\Psi(i,j+2)+\frac{\beta_4}{4\cdot h\cdot k^3}\cdot\Psi(i,j+2)=-f_y(i,j).
\end{gathered}\tag{10}$$

At each point of the finite-difference grid we write an equation like such. In these equations appear the values of Ψ function taken in the neighboring points, resulting in a system of equations to be solved in the $\Psi(i,j)$ nodal values.

For the boundary points, in (10) appear values of Ψ in some non-existing external nodes. These values also appear when we apply (10) for the nodes next to the boundary ones: these external nodes define a new virtual boundary beyond the physical one, increasing the number of the unknowns to be determined.

The system of equations can be solved only by writing boundary conditions: we will give these conditions in all boundary nodes, as prescribed displacements and/or loading forces.

The boundary conditions in the form of prescribed displacement

For easier applicability of this method, let's approximate the physical boundary with one which is made from horizontal and vertical lines adapted to the grid. In this case we define the boundary conditions as the projections of the displacement, as prescribed values of u and/or v. These projections are obtained by deriving of function Ψ, according to the relations:

$$\begin{aligned}
u&=\alpha_1\cdot\frac{\partial^2\Psi}{\partial x^2}+\alpha_2\cdot\frac{\partial^2\Psi}{\partial x\cdot\partial y}+\alpha_3\cdot\frac{\partial^2\Psi}{\partial y^2},\\
v&=\alpha_4\cdot\frac{\partial^2\Psi}{\partial x^2}+\alpha_5\cdot\frac{\partial^2\Psi}{\partial x\cdot\partial y}+\alpha_6\cdot\frac{\partial^2\Psi}{\partial y^2},
\end{aligned}\tag{11}$$

If we express the value of u from the relation (11) with centered differences, we obtain the following equation:

$$\frac{\alpha_2}{4\cdot h\cdot k}\cdot\Psi(i-1,j-1)+\frac{\alpha_3}{k^2}\cdot\Psi(i,j-1)-\frac{\alpha_2}{4\cdot h\cdot k}\cdot\Psi(i+1,j-1)+$$
$$+\frac{\alpha_1}{h^2}\cdot\Psi(i-1,j)-\left(2\cdot\frac{\alpha_1}{h^2}+2\cdot\frac{\alpha_3}{k^2}\right)\cdot\Psi(i,j)+\frac{\alpha_1}{h^2}\cdot\Psi(i+1,j)- \tag{12}$$
$$-\frac{\alpha_2}{4\cdot h\cdot k}\cdot\Psi(i-1,j+1)+\frac{\alpha_3}{k^2}\cdot\Psi(i,j+1)+\frac{\alpha_2}{4\cdot h\cdot k}\cdot\Psi(i+1,j+1)=u(i,j).$$

For v we obtain the same formula, the index of the α -s has to be increased by 3.

We can observe that applying the calculus for a grid node positioned on the boundary, it will be based on three points that are on the imaginary boundary. The concave corners will not raise issues or difficulties, however in the convex corners this scheme would include a point that does not belong to the imaginary boundary (figure 2).

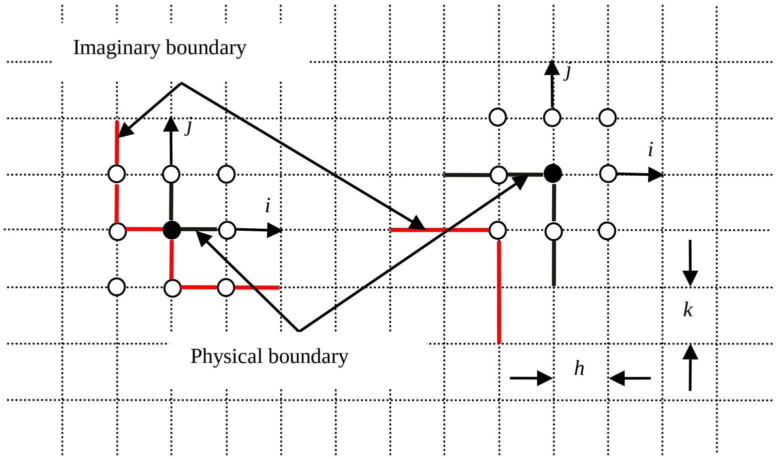

Fig. 2. The imaginary boundary that appears due to finite-difference approximation

In this case instead of centered difference approximation, we apply the derivatives' approximations with the help of forward or backward differences, depending on the corner position (Harangus, 2012).

The boundary conditions in the form of loading

Distributed stress, that loads the boundary is defined by its projections according to x and y directions, noted as p_x and p_y. This stress usually is described by an arbitrary function. During the mesh, this function is replaced by a step function (figure 3).

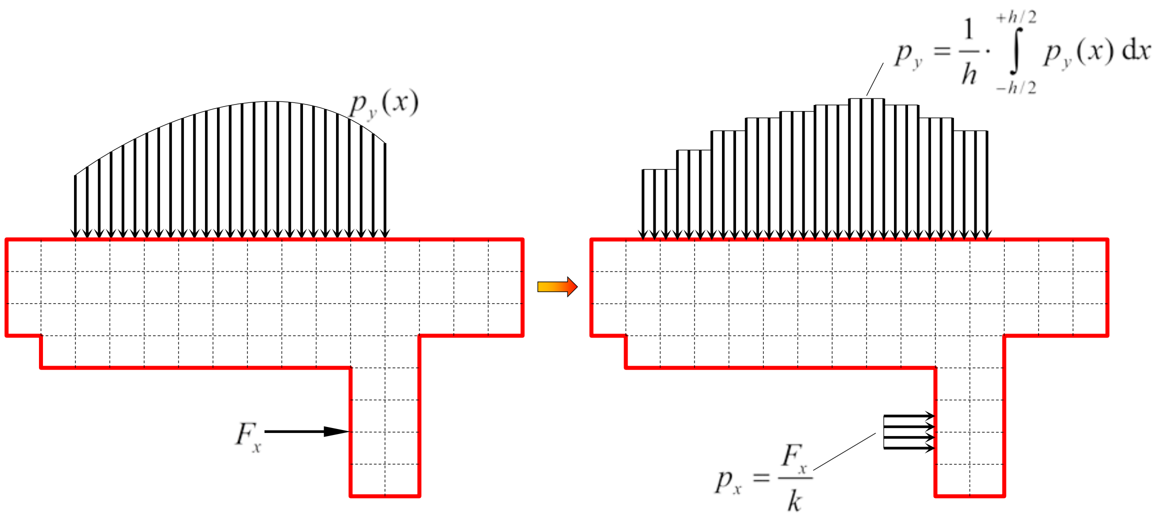

Fig. 3. Task discretion

If we cut an element from the area around a boundary point, the stresses along the boundary must be in balance with the external loads. Therefore we can write the relations which equal the projections of the exterior load with the stresses along the boundary (figure 4).

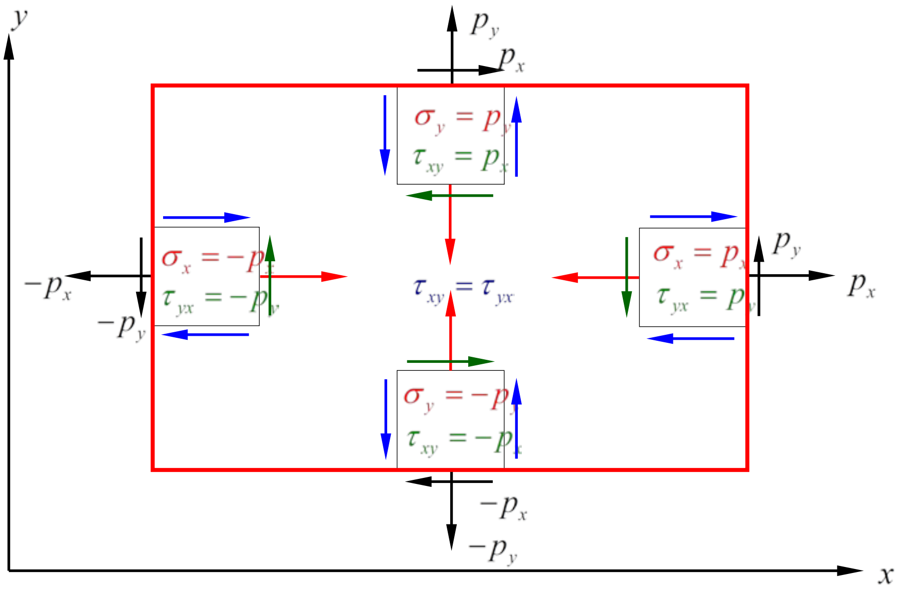

Fig. 4. Stresses along the boundary

Relations between specific stresses and strains are given by the generalized Hooke's law, the specific deformations and displacements of geometrical equations and the displacements and Ψ function by (11) formula. Thus we obtain the relation:

$$\sigma_x = E_{11}\cdot\varepsilon_x + E_{12}\cdot\varepsilon_y + E_{13}\cdot\gamma_{xy} =$$
$$= E_{11}\cdot\frac{\partial u}{\partial x} + E_{12}\cdot\frac{\partial v}{\partial y} + E_{13}\cdot\frac{\partial u}{\partial y} + E_{13}\cdot\frac{\partial v}{\partial x} =$$
$$= E_{11}\cdot\left(\alpha_1\cdot\frac{\partial^3\Psi}{\partial x^3} + \alpha_2\cdot\frac{\partial^3\Psi}{\partial x^2\cdot\partial y} + \alpha_3\cdot\frac{\partial^3\Psi}{\partial x\cdot\partial y^2}\right) +$$
$$+E_{12}\cdot\left(\alpha_4\cdot\frac{\partial^3\Psi}{\partial x^2\cdot\partial y} + \alpha_5\cdot\frac{\partial^3\Psi}{\partial x\cdot\partial y^2} + \alpha_6\cdot\frac{\partial^3\Psi}{\partial y^3}\right) +$$
$$+E_{13}\cdot\left(\alpha_1\cdot\frac{\partial^3\Psi}{\partial x^2\cdot\partial y} + \alpha_2\cdot\frac{\partial^3\Psi}{\partial x\cdot\partial y^2} + \alpha_3\cdot\frac{\partial^3\Psi}{\partial y^3}\right) +$$
$$+E_{13}\cdot\left(\alpha_4\cdot\frac{\partial^3\Psi}{\partial x^3} + \alpha_5\cdot\frac{\partial^3\Psi}{\partial x^2\cdot\partial y} + \alpha_6\cdot\frac{\partial^3\Psi}{\partial x\cdot\partial y^3}\right) =$$
$$= (E_{11}\cdot\alpha_1 + E_{13}\cdot\alpha_4)\cdot\frac{\partial^3\Psi}{\partial x^3} + (E_{11}\cdot\alpha_2 + E_{12}\cdot\alpha_4 + E_{13}\cdot\alpha_1 + E_{13}\cdot\alpha_5)\cdot\frac{\partial^3\Psi}{\partial x^2\cdot\partial y} +$$
$$+(E_{11}\cdot\alpha_3 + E_{12}\cdot\alpha_5 + E_{13}\cdot\alpha_2 + E_{13}\cdot\alpha_6)\cdot\frac{\partial^3\Psi}{\partial x\cdot\partial y^2} + (E_{12}\cdot\alpha_6 + E_{13}\cdot\alpha_3)\cdot\frac{\partial^3\Psi}{\partial y^3}. \tag{13}$$

The boundary conditions are written by stresses with the help of Ψ function derivates, rewriting these derivates with finite-differences.

We exemplify determining conditions of the contour for the vertically side on the left, as follows (Harangus et al, 2012):

$$\begin{aligned}
&-\frac{c_4}{2\cdot k^3}\cdot\Psi(j-2,i)-\left(\frac{c_2}{2\cdot h^2\cdot k}+\frac{c_3}{2\cdot h\cdot k^2}\right)\cdot\Psi(j-1,i-1)+\\
&+\left(\frac{c_2}{h^2\cdot k}+\frac{c_4}{k^3}\right)\cdot\Psi(j-1,i)-\left(\frac{c_2}{2\cdot h^2\cdot k}-\frac{c_3}{2\cdot h\cdot k^2}\right)\cdot\Psi(j-1,i+1)+\\
&+\frac{c_3}{h\cdot k^2}\cdot\Psi(j,i-1)-\frac{c_1}{h^3}\cdot\Psi(j,i)+\left(\frac{3\cdot c_1}{h^3}-\frac{c_3}{h\cdot k^2}\right)\cdot\Psi(j,i+1)-\\
&-\frac{3\cdot c_1}{h^3}\cdot\Psi(j,i+2)+\frac{c_1}{h^3}\cdot\Psi(j,i+3)+\left(\frac{c_2}{2\cdot h^2\cdot k}-\frac{c_3}{2\cdot h\cdot k^2}\right)\cdot\Psi(j+1,i-1)-\\
&-\left(\frac{c_2}{h^2\cdot k}+\frac{c_4}{k^3}\right)\cdot\Psi(j+1,i)+\left(\frac{c_2}{2\cdot h^2\cdot k}+\frac{c_3}{2\cdot h\cdot k^2}\right)\cdot\Psi(j+1,i+1)+\frac{c_4}{2\cdot k^3}\cdot\Psi(j+2,i)=-p_x,
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
c_1 &= E_{11}\cdot\alpha_1+E_{13}\cdot\alpha_4,\\
c_2 &= E_{11}\cdot\alpha_2+E_{12}\cdot\alpha_4+E_{13}\cdot\alpha_1+E_{13}\cdot\alpha_5,\\
c_3 &= E_{11}\cdot\alpha_3+E_{12}\cdot\alpha_5+E_{13}\cdot\alpha_2+E_{13}\cdot\alpha_6,\\
c_4 &= E_{12}\cdot\alpha_6+E_{13}\cdot\alpha_3.
\end{aligned} \tag{15}$$

Conclusions

This paper presents a finite-difference computational method for the integration of differential equations with partial derivatives describing the plane state of displacement or stress of the anisotropic materials. As shown in the paper, the problem can be expressed in stress leading to Airy function, which describes the second order partial differential stress field. With stress and material equations we can determine the specific strains. This method has the disadvantage of the impossibility to express directly the displacements.

By analogy with the Airy function, we used a "potential function" of the displacement, which made it possible to write mixed boundary conditions. The partial derivatives of this function are equal with the displacements in the directions of coordinate axes. Displacement derivatives, as in the derivatives of superior order displacement function give specific strains and by using material equations these superior order derivatives will lead to the stress field. Therefore becomes possible to write outline conditions as distributed load shape, there is a direct relationship (differential equations) between displacements and stresses. These relationships are approximated by finite differences.

In approximation with finite differences the real boundary was replaced by a boundary consisting of horizontal and vertical straight lines and the boundary conditions as prescribed loading led to some equivalence relations between loads and stresses. The denser the grid is, the more accurate the modeling of the load will be and the negative effects of the approximations made in the corner points will be more reduced. The disadvantage of this method is the fact that we can have body forces only in one direction.

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