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On the Philosophy of Probability Behaviour Composite Probability Ƥ(P) In the Case of Discrete Function

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Abstract

This paper is dealing with probability theory, starting from an objectivism philosophic approach, towards mathematical approach. This approach is aiming to study the probability behaviour as a function and as a fraction quantity. This aim, is requiring a discussion of the usage of the probability statement, the causes behind the existence of probabilistic phenomenon, and an explanation that admits to measure the causality by probability theory, through the concept of conditioning and the concept of compliment. This paper is using conceptual experiment with a design that addresses the shortcoming of traditional conceptual experiment. Also, it is using the relative frequency of event and the probability axioms. These, in turn will reflect some aspects of the probabilistic behaviour, in addition to some important consequences that follow from. As a result this will provide a sample space to include all possible events in the form of sequence of sub sequences. Also, this paper will provide some definitions that define some elements of the fractions probabilities and the sample space. These are beside, some proven corollaries that admit to calculate composite probability in the case of discrete function. In addition, to a briefly treatment to the continuous function.

Keywords: Compliment, Experiment, Sample Space, Sequence

Introduction

The theory of probability goes back to Pascal and Fermat and before. But the important period of the development of probability theory was during 1575 to 1825, see (Todhunter, I., 1949). And, for the period of the nineteenth and twentieth centuries, see (Bingham, N. H., 2000). Here, the nature of probability theory has at least two main classes of concepts, see (Parzen, E.,1960). In fact, the concept of probability theory, if it had an ambiguous side, it would be an ambiguous use rather than an ambiguous sense of the probability statement itself. Therefore, this may lead to pose the following question: in what sense the probability statement can be used? A clear literary meaning of a scientific phrase sometimes helps to accept it more in the scientific approach. And, when the statement “probability” carries a determination connotation in its use, without changing the meaning of probability statement, then it will be acceptable more than mere speculation. So, if the need to use probability leads to use the statement of probability and only probability, then the statement of probability is true, otherwise the statement is not necessarily true. Sémantique addresses the changing of word meaning linguistically in details, see (Pierre, Guiraud., 1969). Now, to discuss the question with some logic, there are two conditions; first, if the statement is applied to the occurrence of an absolute event, where the sample space would be unspecified. So, if it means a kind of knowing, then it is not necessarily a probability statement, else the statement would be a non-numerical probability statement. Second, if the occurrence of the event is bounded by several options, such as winning or losing, which means that the sample space is known. So, the statement if it means the unknowing, then it is also not necessarily a probability statement, else it could be a numerical probability statement. Therefore, when the prophet prophesies, he does that with feeling, and he does not depend on a sample space, so he known without sample space, even if that has been taken as a kind of determinism. Such intuition mostly depends on the feeling that constitutes an aspect of its experience. If the statements in the previous case, considered as probability statements and if the probability statement is considered as a kind of knowing, with some conditions, rather than unknowing. Then the knowing will refer numerically to finite or infinite and countability, and the unknowing will refer to uncountability. And they can be studied with the aid of probability theory.

This paper prefers the plural form of the probability word to deal with probability as a set of probabilities. This comes from that, a fraction or part means some, while deterministic may in possible sense mean the whole or the integer. And, because probability does not mean deterministic, it does not mean the whole, but the parts. The causes of the probabilistic phenomenon are what give probability those fractional values. Now, suppose an observer, two events A, B, and the following conditions: If both events had occurred before, and if the event B is only occur when the event A does not occur. Then, the observer has some knowledge about the events A and B and their relations. But, if an event B had never occurred before, and the observer has known nothing about event B, then the occurrence of event A is not always truly deterministic, and the occurrence of event B, falsely would not be probable. The last has a significant point, which is that there is a fraction probability of B occurrence, which does not appear.

Probability Statement Usage

Probability statements are frequently used in political expressions, to avoid determination. Therefore, a single determinism statement could be responded by some probabilistic statements, such as a strategy of game. Now, let $(\mathscr{s}^1)$ denoted to the statement: Probability of an event. When this statement is taken from the point of view of usage, it reflects the meaning of the need to use it. Originally, the probability statement is used to carry the meaning of the occurrence of an event, rather than non-occurrence. But with the mathematical development of probability, the probability statement became abstracted in mathematical formulas. And the concept of probability has acquired the value of 0 to denote the impossible event and 1 to denote the sure event. Thus, in gambling games, each player enters with an incentive probability that favours winning, otherwise the player would not has entered the gambling. For example, if someone y used the statement to expect an event by x probability, then the statement means that y is likely to getting x. So, the probability will take a set of values in the interval; $\left(\frac{1}{2}, 1\right]$ or $\frac{1}{2} < x \leq 1$. And, if y uses it in the opposite case, then the statement means the doubt and the probability will take a set of values in the interval $\left[0, \frac{1}{2}\right)$ or $0 \leq x < \frac{1}{2}$. But, when the word takes the meaning of possible, then its value falls at $\frac{1}{2}$. In either case, some observers $y_1, y_2, \dots, y_n$ will also put their probabilities $x_1, x_2, \dots, x_n$to measure y expectation. Here, probability of the probabilities statement will, in principle, stand out: Statement $(\mathscr{s}^2)$. And there are two ways to evaluate the probability degree, one by whose expecting, and the second by the observers. In addition, when y by his subjective probability predict an event by x, he gives a probability of an inconstant nature that does not carry a determined value. And he will not give a truth without a determined number (degree). So, probability here, need an aspect of determination and probability should be a science that provide a truth. Moreover, if his expectation gives different values of portability for one event in the same time, then he has a mental function f_y that is wrong mathematically. Also, the observers have mental functions $f_{y_1}, f_{y_2}, \dots, f_{y_n}$ that are effected by different factors and would give different values for one variable. In this case, if he gives a probability value equal to φ, then the observers values will be in the range of zero to 1 but for a probability value that does not exceed φ, and this could be expressed by function mapping. Actually, the value of φ would not be determined value but it created and came as discrete quantity.

On the other hand, the concept of probability and its interpretation are connected with epistemology by the function that investigates truth. Take an example of mind (memory), such as example of blind man, if he is knowing people by his individual probability, by their voice, and some one was wrote the names after him; he may wrote a wrong name then a false statement or wrong science. So, probability can be used to measure a truth or certainty. In addition, the statistical probability may carrying determination characteristics compared with the Inductive probability. But, this may be affected by the laws such as religions that are reducing belief in probabilistic nature, or such as state laws that are adopt or do not the probabilistic models.

At the same time, if the intuition gives a deterministic about a certain truth, and proof it theoretically with feasibility to adopt it, the general educational approach that need a certain level with a goal that serve its need for different people, have its effect to adopt or to not adopt certain theory or the intuition itself.

Causality, Conditioning and Compliment

It cannot be asserted that the laws of nature are based on probabilistic model, but it would be feasible to deal with that, they are built on the basis of scarcity and competition. However, it can be said that the nature is built on deterministic laws, but there are causes that lead the results of these laws to have a probabilistic nature. This poses these two questions: Why there is a probabilistic phenomenon? And why probability could measure the causes?

Part of the answer of these questions may carried by this question: is the space able to admit all phenomena to be deterministic?

When tossing a coin, the speculation is the processes of tossing and it is the cause, where probability is the method of calculating or evaluating and it is the result, figure 1.

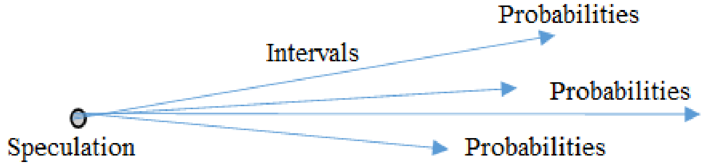

Figure 1. Causes and Results; Speculation-Time-Probability.

But when the processes are doing with a heavy weight, then the processes are not speculation, but it would be determinism processes, relatively. The paradox here, that differs nature from speculation, that the deterministic in nature can take two aspects: to determine a result with one description and one value or to determine the result with two descriptions or more, also with the same one value that distributed over the descriptions, which in turn give it the probabilistic aspect. And, when supposing space, supposing time with an aspect of scarcity. So, some instants may change the case from deterministic phenomenon to be probabilistic phenomenon. Such, in the case of several micro particles with large speed, some factors may carry a kind of scarceness. Bohr has shown in detail in a number of interesting thought experiments how the finite value of the frequently recurring constant ħ in uncertainty relation make the coexistence of wave and particle both possible and necessary. If the probability of finding a particle in some bounded region of space decreases as time goes on, the probability of finding it outside of this region must increase by the same amount. See (Eugen, Merzbacher.,1970).

On the other hand, if the cause is constant cause then it may considered as a constant law and the event may dedicated from universal laws. Else, the event will dedicated from initial conditions. And, if universal laws are considered as conditions then they would be considered as events. So, these conditions take an event from space to space, such that for the conditional probability of an event B, given that an event A has occurred, the sample space reduced from S to A, see (Meyer Paul L., 1970). Mathematically speaking, any law if exist, it is at least meaning a function with known or unknown fixed operations $(\pm, \times, \div, \dots)$, domain and variability with a range. Where in a probabilistic nature, the events may take these paths: Event_1 (cause) gives an event_2 (result). Event_1 consists of event_1, event_2,..., event_i. And, event_2 also consist of event_1, event_2,..., event_j. And there would be a conditioning, so the event may correspond to one or some events. Here, the concept of conditioning can be used and expressed in term of sub σ-field of events. And, in some elementary case, the initial probability space (Ω, ξ, P) is replaced by the probability space (Ω, ξ, P_B), where Ω is the sure event, ξ is σ-field of events and P_B is the conditional probability, see (M. Loève., 1978). But, the condition gives a prior information or event to the last event. And if it synchronize to the last event then it will be a complement and there will be occurrence and non-occurrence. If the phenomenon is a probabilistic phenomenon then it could be resulted by a set of causes that may be created randomly, that is to say, the cause is unknown. This actually pose a question: What the phenomenon and what the cause? If one when he tossing a true coin and just looking for the head when he does not know the existing of the other face (tail), then the phenomenon is that in many thousands trails he getting the head in half of these thousands trails. The unknown face is the cause to get the phenomenon of half, and the two half‘s are in competition nature, so probability is a science of competition. And, in probability theory, if the conditioning is considered as causality for dependent events then this causality could be abbreviated in the concept of compliment for independent events. Conceptually, when probability measures an event compliment, it measures some causes. Also, computation has its causes, which is the scarcity, figure 2.

Scarcity (Variable) → Competition (Variable) → Probabilistic Phenomenon

Figure 2 . Variable Casualty

If the constantly of resulted value is considered as a deterministic result, then the random order will break this deterministic. In the case of throwing 10 true coins by one hand at same time, the causality will be mixed and complicated than the tossing 1 coin. But it will always give a constant frequency.

In general, and in addition to the human boundary knowledge, there are many factors that may lead to the probabilistic phenomenon such as political effectives, moral and religion. Also, there is sometime a human necessity and a nature necessity to have probabilistic phenomena.

Methods

This paper is concern with the concept of probability from a philosophical point of view in an objectivism method, began earlier by some linguistic analysis, as a motivation. But mainly it uses the logical and mathematical analysis to provide mathematical results. And because the conceptual experiment is not impossible to analyse certain phenomena, it will be the followed method, this besides the observation. In addition to some examples which may represent some thought experiments.

Experiment purpose: This experiment is to consist of observing the appearance of fraction heads or fraction tails portability. And to answer these questions: What is the probability of a probability $\wp_i$? And how the probability value behaves?

Starting from the regular experiment of tossing a true coin, its sample space, its probability model and its results. Then proceeding to the following steps:

Step 1: Determining m and choosing n. Then, finding the sample space and building the probability model to find fractions events probabilities.

Step 2: This paper will use a coin, but after shaping it into a spherical shape (true ball), with radius of r. First, dividing curved surface area to find the fraction unit area, which will be: $\mathbb{A} = \left(\frac{4\pi r^2}{mn+m}\right)$ and the area $4\pi r^2$ by some manner will constitute the sample space. But this does not provide a sense until dividing the ball into n ordered closed circles with replacement for every half separately. And, on every circle record a value of a probability fraction, before throwing the ball. Considering the peak of each upper half of the ball as success, figure 2.

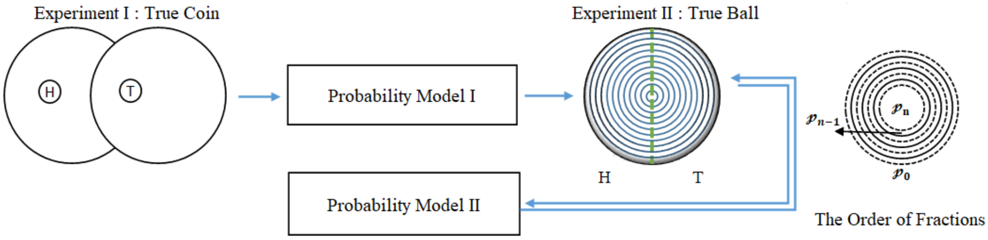

Figure 3. Experiment Model; $(\wp_i)$ value will be non-decreasing value from the middle to the peak

Note, dividing 360° by some numbers, will not always give an integer or finite number. Also, note that the difference between this ball and any celestial body moving in an orbit, is that this ball is moving less freely, so it is either static or moving under the influence of unnatural force. Otherwise, it would have been possible for its details to appear at each point of time t with an approximated determined angle that makes the appearance is known. Moreover, it should be noted that the heavier the weight, the more stable the movement, and vice versa. In addition, one can do this experiment with dice.

Initially, find the probability of head in the experiment of tossing a regular true coin one time (experiment I) as follows:

Definition I: The value $f_A = \frac{m_A}{m}$ is called the relative frequency of event A. And it has the following properties;

$1 - 0 \leq f_A \leq 1$.

$2 - f_A = 0$ if and only if A never occurs, and $f_A = 1$ if and only if A occurs on every repetition.

$3\ -$ If A and B are mutually exclusive events, then $f_{A\cup B} = f_A + f_B$.

The sample space is: $S = \{H, T\}$ and the probability to a head is: $P_H = \frac{H}{S} = \frac{1}{2}$. See (William W. Hines et al.,1990).

Experiment II: First, consider figure 4, that explains the mathematical procedure and the derived expressions.

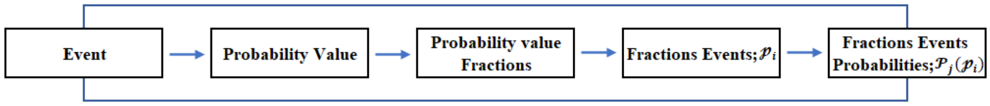

Figure 4. Steps that define fractions events and fractions events probabilities expressions

Also, consider the following definitions:

Definition II: For every event, has a probability value, the fractions of its probability value are events, whatever they are small.

Sample description: This paper will only consider the sample description of experiment I. In addition, the zero fraction occurrence probability is assuming (∅). Moreover, the nil event is the event of non-occurrence. So, the zero fraction of any event is attached with the zero fraction of the other event and being one area or one event.

Definition III: The sample space of experiment II is the space that consists of all the fraction events $\wp_i$.

Now, let, $\mathfrak{D}$ be the sample description or the possible descriptive outcomes of experiment I, m be the quantity of possible descriptive outcomes, $\mathfrak{F}$ be the sample space and $\mathcal{P}_H$ be the probability of an occurrence of the probability value fraction head $\wp_i$ in throwing a true ball once. In addition, consider that all events are independent.

Results

For throwing a true ball there will be $mn + m$ events for each trail. And, for $n = 100$, $\mathfrak{D} = (H, T) = 2 = m$, the number of divisions is $\frac{mn+m}{m} = (n + 1) = 101$ circles, for every ball half, table 1.

Table 1. Sample space for $m \geq 2$

m	$mn + m$ Circles	$\frac{mn+m}{m}$ For Every Part
2	202	101
3	303	101
⋮	⋮	⋮
10	1010	101

And, $\mathfrak{F} = \left(\frac{P(H)}{101}, \frac{P(T)}{101}\right)$ or $\mathfrak{F} = (P(H_0), P(H_1), \ldots, P(H_{100}), P(T_0), P(T_1), \ldots, P(T_{100}))$.

From probability definition if; $\wp_1 + \wp_2 + \cdots + \wp_n = 1$, and if; $\wp_1 = \wp_2 = \cdots = \wp_n$. Then, $n\wp_i = 1$, and $\wp_i = \frac{1}{n}$ for $i = 1, \ldots n$.

Also, $\wp_1 = 1 - (\wp_2 + \cdots + \wp_n), \wp_2 = 1 - (\wp_1 + \wp_3 + \cdots + \wp_n)$, and so on.

Then, $\wp_1 = \wp_2 = \cdots = \wp_n = 1 - [(n-1)\wp_i]$. And, for $\wp_i = \frac{1}{n}$,

$\wp_1 = \wp_2 = \cdots = \wp_n = 1 - \left[(n-1)\frac{1}{n}\right] = 1 - \left[\frac{(n-1)}{n}\right]$ ... (Eq1)

Therefore, by replacing $(n-1)$ by $(n-i)$, there will be an accumulation with replacement, and $\wp_i$ has the following probability values:
$\wp_0 = 1 - (\wp_1 + \cdots + \wp_n), \wp_1 = 1 - (\wp_2 + \cdots + \wp_n), \ldots, \wp_n = 1.$ ... (Eqs 2).
Or, $\wp_0 = 1 - \left[\frac{(n-0)}{n}\right], \wp_1 = 1 - \left[\frac{(n-1)}{n}\right], \ldots, \wp_n = 1 - \left[\frac{(n-n)}{n}\right]$ ... (Eqs 3).
So, the following sequence $\{\wp_n\}$: $\wp_0 < \wp_1 < \cdots < \wp_n$ has the following limit, $\lim_{n\to\infty} \wp_n = \lim_{n\to\infty}\left(1 - \left[\frac{(n-n)}{n}\right]\right) = 1$, and it is convergent.

And, it is included in each other $\wp_0 \subset \wp_1 \subset \cdots \subset \wp_n$.

Now, let, $H = 1, T = 2$, then $\mathcal{P}_H(\wp_i) = \mathcal{P}_1(\wp_i)$ and $\mathcal{P}_T(\wp_i) = \mathcal{P}_2(\wp_i)$.

And here are some terms of $\mathcal{P}_1(\wp_i)$:
$\mathcal{P}_1(\wp_{25} = 0.25) = 2\left[\frac{0.25}{202}\right] = 0.002475248.$
$\mathcal{P}_1(\wp_{100} = 1) = 2\left[\frac{1}{202}\right] = 0.00990099.$
$\mathcal{P}_1(\wp_0 = 0) = 2\left[\frac{0}{202}\right] = 0.$ And, the sum of each series:

$$\sum_{i=0}^{100} \mathcal{P}_1(\wp_i) = m\left[\frac{(1-1)}{\mathfrak{F}} + \frac{(1-0.99)}{\mathfrak{F}} + \cdots + \frac{\left(1-\left(\frac{1}{n}\right)\right)}{\mathfrak{F}} + \frac{(1-0)}{\mathfrak{F}}\right] = \frac{1}{2} = \frac{1}{m}.$$

$$\sum_{i=0}^{100} \mathcal{P}_2(\wp_i) = m\left[\frac{(1-1)}{\mathfrak{F}} + \frac{(1-0.99)}{\mathfrak{F}} + \cdots + \frac{\left(1-\left(\frac{1}{n}\right)\right)}{\mathfrak{F}} + \frac{(1-0)}{\mathfrak{F}}\right] = \frac{1}{2} = \frac{1}{m}.$$

As a result, the sequence associated with the last two finite series is not necessarily a random sequence until it satisfies some conditions, and it could be equidistributed, (k −distributed) or ($\infty -$ distributed), and it depends on a precise definition of random sequence, even if it behaves as a random. Which will depend also on the experiment or phenomenon nature. For more about random sequence and its conditions, see (Knuth, D.,1998).
Table 1, provides the ordered values of $\mathcal{P}_1(\wp_i)$. In this case one may agree that the sample space consists of 101 points. For more about sample space and more applications, see, (Feller, W., 1950).

Table 2 Result of 101 H Fractions

i	$\frac{n-i}{n}$	$1-\frac{n-i}{n}$	$m\left[\frac{\left(1-\frac{n-i}{n}\right)}{\mathfrak{F}}\right]$	i	$\frac{n-i}{n}$	$1-\frac{n-i}{n}$	$m\left[\frac{\left(1-\frac{n-i}{n}\right)}{\mathfrak{F}}\right]$	i	$\frac{n-i}{n}$	$1-\frac{n-i}{n}$	$m\left[\frac{\left(1-\frac{n-i}{n}\right)}{\mathfrak{F}}\right]$	i	$\frac{n-i}{n}$	$1-\frac{n-i}{n}$	$m\left[\frac{\left(1-\frac{n-i}{n}\right)}{\mathfrak{F}}\right]$
0	1	0	0	25	0.75	0.25	0.002475248	50	0.5	0.5	0.004950495	75	0.25	0.75	0.007425743
1	0.99	0.01	9.90099E-05	26	0.74	0.26	0.002574257	51	0.49	0.51	0.005049505	76	0.24	0.76	0.007524752
2	0.98	0.02	0.00019802	27	0.73	0.27	0.002673267	52	0.48	0.52	0.005148515	77	0.23	0.77	0.007623762
3	0.97	0.03	0.00029703	28	0.72	0.28	0.002772277	53	0.47	0.53	0.005247525	78	0.22	0.78	0.007722772
4	0.96	0.04	0.00039604	29	0.71	0.29	0.002871287	54	0.46	0.54	0.005346535	79	0.21	0.79	0.007821782
5	0.95	0.05	0.00049505	30	0.7	0.3	0.002970297	55	0.45	0.55	0.005445545	80	0.2	0.8	0.007920792
6	0.94	0.06	0.000594059	31	0.69	0.31	0.003069307	56	0.44	0.56	0.005544554	81	0.19	0.81	0.008019802
7	0.93	0.07	0.000693069	32	0.68	0.32	0.003168317	57	0.43	0.57	0.005643564	82	0.18	0.82	0.008118812
8	0.92	0.08	0.000792079	33	0.67	0.33	0.003267327	58	0.42	0.58	0.005742574	83	0.17	0.83	0.008217822
9	0.91	0.09	0.000891089	34	0.66	0.34	0.003366337	59	0.41	0.59	0.005841584	84	0.16	0.84	0.008316832
10	0.9	0.1	0.000990099	35	0.65	0.35	0.003465347	60	0.4	0.6	0.005940594	85	0.15	0.85	0.008415842
11	0.89	0.11	0.001089109	36	0.64	0.36	0.003564356	61	0.39	0.61	0.006039604	86	0.14	0.86	0.008514851
12	0.88	0.12	0.001188119	37	0.63	0.37	0.003663366	62	0.38	0.62	0.006138614	87	0.13	0.87	0.008613861
13	0.87	0.13	0.001287129	38	0.62	0.38	0.003762376	63	0.37	0.63	0.006237624	88	0.12	0.88	0.008712871
14	0.86	0.14	0.001386139	39	0.61	0.39	0.003861386	64	0.36	0.64	0.006336634	89	0.11	0.89	0.008811881
15	0.85	0.15	0.001485149	40	0.6	0.4	0.003960396	65	0.35	0.65	0.006435644	90	0.1	0.9	0.008910891
16	0.84	0.16	0.001584158	41	0.59	0.41	0.004059406	66	0.34	0.66	0.006534653	91	0.09	0.91	0.009009901
17	0.83	0.17	0.001683168	42	0.58	0.42	0.004158416	67	0.33	0.67	0.006633663	92	0.08	0.92	0.009108911
18	0.82	0.18	0.001782178	43	0.57	0.43	0.004257426	68	0.32	0.68	0.006732673	93	0.07	0.93	0.009207921
19	0.81	0.19	0.001881188	44	0.56	0.44	0.004356436	69	0.31	0.69	0.006831683	94	0.06	0.94	0.009306931
20	0.8	0.2	0.001980198	45	0.55	0.45	0.004455446	70	0.3	0.7	0.006930693	95	0.05	0.95	0.009405941
21	0.79	0.21	0.002079208	46	0.54	0.46	0.004554455	71	0.29	0.71	0.007029703	96	0.04	0.96	0.00950495
22	0.78	0.22	0.002178218	47	0.53	0.47	0.004653465	72	0.28	0.72	0.007128713	97	0.03	0.97	0.00960396
23	0.77	0.23	0.002277228	48	0.52	0.48	0.004752475	73	0.27	0.73	0.007227723	98	0.02	0.98	0.00970297
24	0.76	0.24	0.002376238	49	0.51	0.49	0.004851485	74	0.26	0.74	0.007326733	99	0.01	0.99	0.00980198
												100	0	1	0.00990099
												101	$\sum_{i=0}^{100} \mathcal{P}_H(P_{v_i})$		0.5000

To generalize let $\mathcal{P}_j(\wp_i)$ for $m \geq 2$, and $j = 1,2, \dots m$, m is a positive integer number, i and n is initially a positive integer number, but it could be extended to take a positive rational number Q. And, for i to take a negative value, it should be in absolute.

Now, for $j > 1$, $\wp_i = 1 - \left(\frac{jn-ji}{jn}\right)$ and is considered as a distribution for $(P(H), P(T))$ or $(\frac{1}{2}, \frac{1}{2})$ mass in some fractions n and this distribution provides a description of $\wp_i$ behaviour, see (Larson, Harold J.,1973).

Now, $\mathcal{P}_j(\wp_i) = m\left[\frac{1-\left(\frac{jn-ji}{jn}\right)}{\mathfrak{F}}\right]$. So, $\{\mathcal{P}_j(\wp_i)\}$ is the sequence of the following probability values:
$\mathcal{P}_1(\wp_0, \dots, \wp_n), \mathcal{P}_2(\wp_0, \dots, \wp_n), \dots, \mathcal{P}_m(\wp_0, \dots, \wp_n)$.

And has the following limit, $\lim_{\substack{n\to\infty \\ m\to\infty}} \mathcal{P}_m(\wp_n) = \lim_{\substack{n\to\infty \\ m\to\infty}} \frac{m}{mn+m}\left[1 - \left(\frac{mn-mn}{mn}\right)\right] =$

$$\lim_{n\to\infty} \frac{1}{n+1}\left[1-\left(\frac{n-n}{n}\right)\right] = \lim_{n\to\infty}\left(\frac{1}{n+1}\right) - \lim_{n\to\infty}\left(\frac{n-n}{n^2+n}\right) = \lim_{n\to\infty}\left(\frac{1}{n+1}\cdot\frac{n+1}{n+1}\right) - 0 = \lim_{n\to\infty}\left(\frac{(n+1)}{(n+1)(n+1)}\right) = \lim_{n\to\infty}\left(\frac{(\frac{n}{n}+\frac{1}{n})}{(\frac{n}{n}+\frac{1}{n})(\frac{n}{n}+\frac{1}{n})}\right) = \lim_{n\to\infty}\left(\frac{(1+\frac{1}{n})}{(1+\frac{1}{n})(1+\frac{1}{n})}\right) = 1,$$ and it is convergent.

Now, $\sum_{j=1}^{m}\sum_{i=1}^{n}\mathcal{P}_j(\wp_i) = [\mathcal{P}_1(\wp_0)+\mathcal{P}_1(\wp_1)+\cdots+\mathcal{P}_1(\wp_n)] + [\mathcal{P}_2(\wp_0)+\mathcal{P}_2(\wp_1)+\cdots+\mathcal{P}_2(\wp_n)] + \cdots + [\mathcal{P}_m(\wp_0)+\mathcal{P}_m(\wp_1)+\cdots+\mathcal{P}_m(\wp_n)] = 1$.

And, the domain of $\mathcal{P}_j(\wp_i)$ is the set of real numbers in the closed interval $[0\,,1]$, and the range of the $\mathcal{P}_j(\wp_i)$ is the set of real numbers in the closed interval $\left[0,\frac{m}{mn+m}\right]$, for all j, table 3.

Table 3. $\wp_i$ Range is the Domain of $\mathcal{P}_j(\wp_i)$ with Some passable intervals

<table>
<tr><td colspan="2">$\wp_i$</td><td></td></tr>
<tr><td>Domain</td><td>Range</td><td>$\mathcal{P}_j(\wp_i)$</td></tr>
<tr><td rowspan="2">$[0,\infty)$, $(-\infty,\infty)$ And, $n>0$</td><td>Domain</td><td>Range</td></tr>
<tr><td>$(0,1]$, $(0,1)$, $[0,1]$ or $[0,1)$</td><td>$\left(0,\frac{1}{mn+m}\right]$, $\left(0,\frac{1}{mn+m}\right)$, $\left[0,\frac{1}{mn+m}\right]$ or $\left[0,\frac{1}{mn+m}\right)$</td></tr>
</table>

As a result of dividing the ball into n ordered circles with replacement for every half separately, there will be inequal ordered division. And, as a primary condition to implement the experiment II, consider the following proposition:

Proposition I: The greater the fraction probability value, the greater the probability of its occurrence. Such that, the occurrence of $\wp_n$ implies the occurrence of $\wp_{n-1},\dots,\wp_0$, but the occurrence of $\wp_{n-1}$ does not implied the occurrence of $\wp_n$, and so on.

Proof: Suppose that,

$$I_r := \left[0,\frac{1}{r}\right], I_{r-1} =: \left[0,\frac{1}{r-1}\right], \dots, I_{r-(r+1)} =: \left[0,\frac{1}{r-(r-1)}\right], r \neq 0.$$

Are some intervals on the real line, $l_1, l_2, \dots, l_k$ are the lengths of these intervals, where k and r are positive integers, and $\mathfrak{p}_0, \mathfrak{p}_1, \dots, \mathfrak{p}_n$ are some sets belong to these intervals, respectively. And, if $I_r \subset I_{r-1} \subset \cdots \subset I_{r-r+1}$ and $l_1 < l_2 < \cdots < l_k$, then $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$, and if $\wp_0 \in \mathfrak{p}_0, \dots, \wp_n \in \mathfrak{p}_n$. Then, $\wp_0 < \wp_1 < \cdots < \wp_n$. And this proposes that: If this is true for the sub sequence $\{\wp_i\}$ of the sequence $\{\mathcal{P}_j\}$, then it is true for the sequence $\{\mathcal{P}_j\}$. To clarify the theoretical idea behind this proposition, suppose the experiment of tossing true coin in table 4.

Table 4. The probability of H is greater than the probability of T in the interval $(t_2, t_4]$

Toss	Result	Equality	Time			
1	T	T > H	t_1			
2	H	T = H		t_2		
3	H	T < H			t_3	
4	H	T < H				t_4

Definition IV: The probability of probabilities is a composite function.

Let $\wp$ and $\mathcal{P}$ are some functions such that $\wp$: $D_1 \to R_1$, and $\mathcal{P}: R_1 \to R_2$ respectively, then $(\mathcal{P} \circ \wp)$ is a composite function. To illustrate that, let $x = (n - i)$, $\wp(x) = 1 - \frac{x}{n}, \mathcal{P}(x) = \frac{x}{n+1}$, for $x \leq n$, and $x, n \in R$.

Then, the composite function is: $\mathcal{P}(x) = \frac{1-\frac{x}{n}}{n+1} = \frac{n-x}{n(n+1)}$.

Generally, if x is a real number and X is a discrete random variable (real valued function), then, the distribution function is:

Let, x denote the number of occurrences of $\wp_i$. Then,
$F_X = P_X(X \leq x)$, for all $x \in (-\infty, \infty)$ and the probability function is:
$P_X(x_i) = F_X(x_i) - F_X(x_{i-1})$.
$F_X(x_i) = P_X(X \leq x_i) = \sum_{x \leq x_i} P_X(x)$.

And, the probability distribution of P_x is the collection of pairs; $\left[(x_i, P_X(x_i), i = 1,2, \dots\right]$. See (William W. Hines et al.,1990). And, for the possible graphs of the distribution function and its characteristics, see (Edward, J. Dudewicz., 1988). On the other hand, the density function could be divided into m parts, for example in the case of continuous random variable if $\mathcal{P}_1(\wp_0, \dots, \wp_n), \mathcal{P}_2(\wp_0, \dots, \wp_n), \dots, \mathcal{P}_m(\wp_0, \dots, \wp_n)$, are nonnegative probability values that sum to 1, and if the interval $(0, t]$ divided into m parts. Then the density function will divided into m parts, and $F_1, F_2, \dots, F_m$ are some distributions. For the techniques, see (Knuth, D.,1998).

Definition V: For every fraction event, there is a probability of its occurrence and denoted by $\mathcal{P}_j(\wp_i)$.

Corollary I: Every probability is consist of finite or countable infinite set of probabilities and every independent event consists of fractions events $\wp_i = 1 - \left(\frac{n-i}{n}\right)$. Here, the index of $\wp$ could be extended to $\mathcal{P}_j(\wp_{i_{1,2,\dots,l}} \dots)$. See (Borel, Émile., 1898), and (Khinchin, A. Ya., 1997).

Now, the following proves that $\mathcal{P}_j(\wp_i)$ is verifying the probability axioms:

Proof :

1- As all values of $\mathcal{P}_j(\wp_i)$ are not negative, then $\mathcal{P}_j(\wp_i) \geq 0$.

2- If $\mathcal{P}_j(\wp_i) = \frac{f_{\wp_i}}{n}$, where $f_{\wp_i}$ is the frequency, and n is an integer number such that, $f_{\wp_i} \leq n$, and $\frac{f_{\wp_i}}{n} \leq 1$, then $\mathcal{P}_j(\wp_i) \leq 1$.

3- To prove that, $\mathcal{P}_j(\mathfrak{F}) = 1$. Suppose that, $\mathfrak{F} = \{H\}$ then, $P(\mathfrak{F}) = P(H) = 1$, and for $\mathcal{P}_j(\mathfrak{F})$ and $m = 1$, let $\sum_{i=0}^{n} \mathcal{P}(\wp_i) = \mathcal{P}(\mathfrak{F})$ and $\sum_{i=0}^{n} \mathcal{P}(\wp_i) = \frac{1}{(n+1)} \sum_{i=0}^{n} \wp_i = 1$ then, $(n+1)\wp_i = (n+1) = 1 = \mathcal{P}(\mathfrak{F})$. Hint, for $\mathfrak{F} = \{H\}$, suppose tossing a coin with H on its both sides, once.

Also, the countable infinite property can be proved intuitively in figure 4.

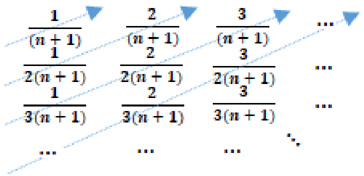

Figure 5. The Set of $\mathcal{P}_j(\wp_i)$ Infinite Elements

Lemma I : For every independent event E, the compliment event probability $P(\bar{E})$ is the fractions events of that event if and only if $P(E) = P(\bar{E})$.

Proof: Let E_1, E_2 are some independent events and $\wp_i$ be the fractions events of $P(E_1)$. And, if $P(E_1) = P(E_2)$, then $P(E_1) = P(\bar{E}_1)$ then, the fraction of $P(E_1)$ is $\wp_i = \frac{P(E_1)}{n}$ and, $\wp_i = \frac{P(\bar{E})}{n}$, and if $n = 1$ then $\wp_i = P(\bar{E})$.

Hint, for $n > 1$ and $m > 2$ take $\sum_{i=0}^{n} \wp_i$ and note that if $E \cap \bar{E} = \emptyset$, then $P(\bar{E}_1) = P(E_2) \cup \ldots \cup P(E_m)$.

Corollary II : For every fraction event $\wp_{j_i}$, a compliment event $\bar{\wp}_{j_i}$ is, $\bar{\wp}_{j_i} = \wp_{j_n} - \wp_{j_i} + \wp_{l \neq j,\ldots,m_n}, i = 0 \ldots., n$, and $j, l = 1, \ldots, m$, if and only if $\wp_{j_i}$ is satisfying proposition I. And, every fraction event $\wp_i$ is a fraction of a fraction event $\wp_{i+1}$ if and only if $\wp_i < 1$. Proof: See, Eqs 3.

Intuitively, set is some different elements with common features or certain source, and every set to be construct, it is constructing by ϕ, this is beside some axioms. Such that, for some sets A and B with only one element and without any common elements, then $A \cap B = \emptyset$, if $a \in A$ and $b \in B$, and $(a \neq b)$. So, if $(a = a$ or $a = \emptyset)$ then $A \cap B = \emptyset$, at the same time, if $(a \neq a$ and $a \neq \emptyset)$ or if $(b \neq b$ and $b \neq \emptyset)$ then at least, one of the two sets is not exist. So, every set to be exist it should consist of at least $\emptyset$. Thus, there exist an empty set with $\emptyset$ of some features (or description) and without any element. And from set theory, a power of set could be 2^N and if a sample space is a set of events, then there are 2^N subsets, where N is a finite size for a sample description space, see (Parzen E., 1960). So, in the present case for $m = 2$, and for throwing the ball twice, there will be $2^{m^2n^2}$ subsets after excluding $\mathcal{P}_1\wp_0$ and $\mathcal{P}_2\wp_0$, which already represented by ϕ in $\mathfrak{F}_{\mathfrak{F}}$.

Such that, $((P(H_0), P(T_0)): \wp_{H_0}, \wp_{T_0}: \emptyset)$. And,
$((\wp_{H_1}, \wp_{T_1}), \dots, (\wp_{H_{100}}, \wp_{T_{100}}), (\wp_{T_1}, \wp_{H_1}), \dots, (\wp_{T_{100}}, \wp_{H_{100}}),$
$(\wp_{H_1}, \wp_{H_1}, \dots, (\wp_{H_{100}}, \wp_{H_{100}}), (\wp_{T_1}, \wp_{T_1}, \dots, (\wp_{T_{100}}, \wp_{T_{100}}).$
And, if $\mathfrak{F}_1 = (\wp_{H_1}, \wp_{T_1}), \mathfrak{F}_2 = (\wp_{H_1}, \wp_{T_2}), \mathfrak{F}_3 = (\wp_{H_1}, \wp_{T_3}), \dots$, and so on.
Then, $\mathfrak{F}_{\mathfrak{F}} = \{\{\mathfrak{F}_1\}, \{\mathfrak{F}_2\}, \{\mathfrak{F}_3\}, \{\mathfrak{F}_4\}, \dots, \{\mathfrak{F}_1, \mathfrak{F}_2\}, \dots, \mathfrak{F}, \phi\}$.

Now, if $\mathfrak{F}$ is the fundamental set of events.
Let, A and B any two events of $\mathfrak{B}$ ($A \in \mathfrak{B}, B \in \mathfrak{B}$). Then, $A \cup B \in \mathfrak{B}$, the compliment $\bar{A} \in \mathfrak{B}$ and $A \cap B \in \mathfrak{B}$. Then, a nonempty collection of subsets $\mathfrak{B}$ of a set $\mathfrak{F}$ is a σ-field of subsets of $\mathfrak{F}$.

Definition VI: A probability measure $\mathcal{P}(\cdot)$ on a σ-field of subsets $\mathfrak{B}$ of a set $\mathfrak{F}$ is a real valued function having domain satisfying the following properties:
$1 -\ \mathcal{P}_j(\wp_i) \geq 0$ for all $\wp_i \in \mathfrak{F}$.
$2 - \mathcal{P}_j(\mathfrak{F}) = 1$.
$3 -$ If $\wp_i, i = 1,2,3$ ... are mutually disjoint sets in $\mathfrak{F}$ then,
$\mathcal{P}_j(\cup_i^\infty \wp_i) = \sum_j^m \sum_i^\infty \mathcal{P}_j(\wp_i)$.

And, the probability space is:$(\mathfrak{F}, \mathfrak{B}, \mathcal{P}(\cdot)\,)$. See (Paul G. Hoel et al.,1971).

Here, $\mathcal{P}(\cdot)$ could be defined by different functions, which may give different values. But, this will depend on the explanation of experiment, sample space and the description of the event, see (Parzen, E.,1967). Also, these and the function behaviour itself could reflect some aspects of probability values behaviour.

Corollary III: The impossible occurrence happen if and only if the occurrence probability of all events are equal to $\frac{1}{m}, m \geq 2$, simultaneously. (Here the peak point of every event supposed to be $\wp_n$).

Proof: If $\frac{1}{m}$ is equal to the upper half area of ball, then the occurrences of $\mathcal{P}_{j=1,\dots,m}\wp_{i,\dots,n}$ are happen only when $\mathcal{P}_{\substack{l=1,\dots,m \\ l\neq j}}\wp_{i,\dots,n}$ are not happen.

As a result, $\mathcal{P}_{\substack{l=1,\dots,m \\ l\neq j}}\wp_{i,\dots,n} = 0$. And, if
$\mathcal{P}_{j=1\dots,m}\wp_{i,\dots,n} = \mathcal{P}_{\substack{l=1,\dots,m \\ l\neq j}}\wp_{i,\dots,n}$, simultaneously, then $\mathcal{P}_{j=1,\dots m}\wp_{i,\dots,n} = 0$.

So, there is no an occurrence, and this pose the following:

Corollary IV: The nil (equally fractions) occurrence happen, if and only if the occurrences probabilities of fractions events of all or some events are equal to $\frac{1}{2m}$ simultaneously.

Proof: It is analogous to the previous proof and has the same result. But the difference here, is that the peak point of all or some events will be zero or $\wp_0$ simultaneously, and here there is a representation of $\emptyset$.

Corollary V: For more than one continuous functions of independent event, if one of them is non-decreasing function at time t in the half interval $(o, t]$ then the other one is necessarily decreasing function at t.

Proof : Take $m = 2$ and suppose that, $\mathcal{P}_1(\wp_i) > \mathcal{P}_2(\wp_i)$ and $\frac{1}{2} < \mathcal{P}_1(\wp_i) < 1$.

Then, from $\mathcal{P}_1(\wp_i) + \mathcal{P}_2(\wp_i) = 1$ and $\mathcal{P}_2(\wp_i) = 1 - \mathcal{P}_1(\wp_i)$. And, from

$\mathcal{P}_1(\wp_i) > \mathcal{P}_2(\wp_i)$ then $\mathcal{P}_1(\wp_i) > 1 - \mathcal{P}_1(\wp_i)$. "Continuous to right."

And it is clear that every increasing in $\mathcal{P}_1(\wp_i)$ gives a decreasing in $\mathcal{P}_2(\wp_i)$, if and only if these functions are continuous functions in the open interval $(\frac{1}{2}, 1)$.

So, the function behaviour reflects the probability values behaviour of these two events.

Finally, the construction of $\mathcal{P}_j(\wp_i)$ is not restricted to the coin experiment. But it could be constructed for any random experiment, such as Brownian motion for two independent continuous parameter random variables $X_d(t)$ and $X_l(t)$, where $\{X(t), 0 \leq t < \infty\}$.

Discussion

What are the deterministic aspects that probability theory can carry? This includes the description set that the probability theory provides by specific sample space, this in addition to the finite and infinite countable aspects.

How the statement of "probability of the probabilities" could be carried out? A probability of the probabilities statement ($\mathcal{s}^2$) will take an analogous concept of the mathematical statements of log $(\log x)$ or $\sqrt{\sqrt{x}}$, and to employ the statement of $\mathcal{s}^2$ in same function of the statement $\mathcal{s}^1$. Moreover, the statement $\mathcal{s}^2$ has derived from statement $\mathcal{s}^1$, not merely to measure the statement $\mathcal{s}^1$, but to provide a necessary meaning of the statement $\mathcal{s}^1$, and not as a mathematical necessity, but as a conceptional necessity.

Why the fractions are unequal in the experiment design? It is necessary for the fractions to have an ordered probability values to be unequal, and this happen because every fraction had been taken with replacement. But the two faces are equally likely. Also, this unequal ordered can provide a description for the moments of $\mathcal{P}_H(\wp_i)$ or $\mathcal{P}_T(\wp_i)$ tendency to the value of $\frac{m}{mn+m}$.

Is it possible to distribute fractions events randomly on the ball? And, what is the reasonable criteria to choose n divisions? It is possible and useful in some cases to distribute the fractions events with a random topological extent in direction and curvatures, but in this case the uniformity of head or tail will be impossible. And, the main criteria to choose n divisions is to answering the question of, what the required fraction quantity that can change event occurrence in prior to last step, what ever it is small.

Is there a natural necessity in the experiment of tossing a true coin? Here, both events should be considered as one event and the natural necessity, if exists, will be a common. At the same time, there will be missing events, and this does not satisfy the natural necessity condition. For more discussion about physical experiment see (Werner, Heisenberg., 1930) and for the definition of experiment see (Rényi, Alfred., 1970).

How to explain zero fraction? As there is no physical fraction has zero value, then the probability of zero is zero. But when zero is considered as a fraction, then there is no an occurrence but the sure occurrence of a fraction which is the zero. And if there is need to study the nil event, consider zero as a fraction. Moreover, in the case of true coin, if the coin lands on the edge, the probability will not be $P(H) = P(T) = \frac{1}{3}$, unless considering the weight, see (Howard M. Taylor et al.,1998). So, the dividing line between events, whatever it is small, can be considered as an non-occurrence, and this non-occurrence is necessary (inevitable).

Mathematically, how the result uses the probability theory? The result used the probability axioms through a composite function in several sequences for discrete sample space and admits the sample space to include all possible fractions events. Also, if the ball continues to move randomly by throwing it once on an unlevelled surface and observing $\wp_i$ on the peak of the ball at every time t until the ball stops completely. Then the function will be continuous at time t. Moreover, all neighborhood fractions to the peak could take different measures. And have the sense of distance from the peak, or the distance between two events. In some physical problems, the probability space constructing will depends on the questions: Is there a σ-field $\mathfrak{B}$ that contains all intervals as members. And a probability measure defined on $\mathfrak{B}$, that assigns the desired probability to the interval. See (Paul G. Hoel.,1971), (Feller W., 1971) and (Tucker, Howard G., 1967). But for processes such as centered Gaussian processes “fractional brownian motion” with advanced treatment see (W. V. Li et al.,2001).

Is the result repeats the concept of distribution function or the concept of the probability sum? The distribution function tells how the values of the random variable are distributed, and gives the distribution of values in cumulative form. See (Alexander, M. Mood et al., 1974). But the resulted function is a non-decreasing function, and only the $\{\wp_i\}$ sequence behave as a distribution function. Also, the result does not interpret the fractions of the probability into a total probability, but rather seeks to interpret the probability as a set of fractions probabilities. At the same time, it is not partitioning set into a subset, even if it uses partitioning techniques.

How does the result differ from the frequency theory of Von Mises? The result differs in two aspects: First; the experiment aspect. Second; the result gives a value to the probability of probabilities and does not proceed from that each event has a certain character. See (Popper, K.,2002), and (Richard Von Mises., 1957).

Conclusion

This paper concluded that, the probability statement sense is depending on the necessity to use it, and this necessity is also depending on some factors, such as the level of knowledge or culture. Moreover, using of probability theory is connecting with epistemology as an approach or intuition to investigate truth, and as a science to measure or to provide a truth. In addition, this paper concluded that the notion of one-to-one correspondence is employed as a natural necessity condition in the occurrence of some events, where one and only one event happen among all events at a certain time t. So, there will be an event, time, and a unique natural necessity condition. And this condition would not be unique condition without the fraction element.

Significantly, this paper concluded that the probability behavior can interpreted as a discrete quantity consists of coherent quantities, which admits calculating probability by probability. So, for the event to be occur, the fractions probabilities should be discrete quantity of some coherent quantities, because there is a cumulative $\wp_i$ that distributed on every ball half. Also, this can reflect the dynamic part of the probability theory. Which in turn can describe the probability values behavior rather as a stochastic processes.

Regarding causality, this paper concluded that, the most important cause behinds the existence of probability phenomenon is the cause of scarceness that motivate the concept of competition. This scarceness has an influential, that impact an event or more to happen, and to create a probabilistic phenomenon. In addition, if the zero fraction and dividing line are considered as fractions, then the fractions appearance is determinism by the cause of throwing the ball, but what is not determinism is the appearance of the head or the tail face. And, from statistical point of view, and from the probability values behaviour, this paper concluded that when a particle's position is measured with a probabilistic aspect, through an ideal experiment in constant conditions. Then, the existence of the particle at that position could be discussed from point of view of a fraction of an occurrence, which also effected by the replacement law.

From experiment, this paper concluded that, the experiment designing should be significant to achieve the nature logic of the occurrence, which in turn admits to study the behaviour of probability values. Also, it admits to searching for a presence of another possible outcome. And that, the method of designing the experiment eliminates the inevitability of the coin falling on one of its two sides completely, which is one of its defects that ignores the other partial possibilities. So the fall should be less restrained, as nature is tend to distributed toward optimal fair shape. As a result, this experiment represents physical body fractions theoretically by a probability value that considering the physical body existence as a portability, that take value from 0 to 1, which mathematically are existed values. Moreover, this experiment is able to represent ϕ by 0 and it is able to represent all probability values, that are less than 1. Also, this experiment is able to represent a pairwise occurrence $(\mathcal{P}_1(\wp_i)\,,\mathcal{P}_2(\wp_i))$ if the upper half of the ball (instead of the peak), is considered as the success. This in addition, to the concept of sample space which extended to be $mn + m$ outcomes.

Regarding the composite function $\mathcal{P}_j(\wp_i)$, it is expressed in term of sequences $\{\mathcal{P}_j(\wp_i)\}$ and sub sequence $\{\wp_i\}$. Both have probabilistic values. And, all probability fractions will distributed by $\wp_i$ as follow $\wp_i = \wp_{i-1} + \wp_0$ for $i \geq 2$. Such that, the occurrence of $\wp_2$ included the occurrence of $\wp_1$. In addition, the sequence $\{\mathcal{P}_j(\wp_i)\}$ can be considered as a random sequence if there exist an infinite random sequence $\{\mathcal{P}_{1,2,..+\infty}(\wp_{1,2...+\infty})\}$, and it is $(\infty -$ distributed) in the interval $[0,1)$. In addition, the outcome of the experiment is described by a sample space $\mathfrak{F}$ with a probability function $\mathcal{P}(\cdot)$ of a discrete random variable and the corresponding distribution function.

From lemma I, corollary I and II, this paper concluded that the compliment event probability could be the fractions events of an event, and every event could be treated as fractions events. And such, the probability theory can measure causes by measuring the conditioning and complement events. Also, the compliment of any occurrence is the non-occurrence or the cause, if it is exist. Also, from corollary IV, an event will not occur, and it will not acquire its probability of occurrence, until more than $\frac{1}{2m}$ of its fraction's occurred.

Conflicts of Interest: None declared.

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