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The Philosophy of Probability Values Behaviour, through Fractions and Composite Probability Function in the Continuous Case

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Abstract

This paper deals with probability theory, and it is an extension to a published paper that has the same title, but for the discrete case. This present paper is aiming to study probability values behavior, in the case of continuous sample space, through fractions intervals and composite function. This aim tends to study the value behavior rather than finding the value itself. Also, this aim requires usage of some concepts of continuity, geometric probability, and measure theory, which also need a brief treatment. This paper is mainly using an experiment with a design that helps to study the probability fractions values in the form of intervals in the case of one direction movement and in the case of different directions. As a result, every case reflects some aspects of probability values behavior and can clarify many important characteristics of the probability theory. In addition to applying the composite function by some important theorems of conditional probability. These are besides a proven proposition that helps to design experiment, upon the understanding of the case nature. In addition to a corollary that allows to visualize negative probability values, as a particular case (trial), that upon the validity of the explanation of negativity which should be consistent with the probability axioms.

Keywords: Geometric Probabilities, Negative Probabilities, Dependent

Introduction

The difference between haphazard, chance or accident, and the act is a thin thread. Such as the difference between finding a dumped coin and the chance between two players. The difference is that in the game of chance, every player has some information about the other and has his influence. But, in the first case no one has an idea about who may find the coin, in addition to the independence factor. In randomness, events happen without any intentional action, and it is difficult to be predicted, also it is rarely repeated, in addition to the factor of homogeneity. Laplace said : Les événements actuels ont avec les précédents une liaison fondée sur le principe évident, qu'une chose ne peut pas commencer d'être, sans une cause qui la produise. Cet axiome, connu sous le nom de *principe de la raison suffisante*, s'étend aux actions mêmes que l'on juge indifférentes. La volonté la plus libre ne peut, sans un motif déterminant, leur donner naissance ; car si, toutes les circonstances des deux positions étant exactement semblables, elle agissait dans l'une et s'abstenait d'agir dans l'autre, son choix serait un effet sans cause : elle serait alors, dit Leibnitz, le hasard aveugle des épicuriens. L'opinion contraire est une illusion de l'esprit qui, perdant de vue les raisons fugitives du choix de la volonté dans les choses indifférentes, se persuade qu'elle s'est déterminée d'elle-même et sans motifs.

On the other hand, the continuous concept has a key role in randomness. In addition, in the probabilistic nature, the discrete process is usually part of a continuous process, and it appears if the continuous process has interrupted. Moreover, when human consciousness or technology ability cannot recognize the entire process or the uncountable instant such as the absolute speed, then there is a need to consider it as a continuous process. So, the unseen are the events that happened as discrete events in a continuous process or as an unrepeated discrete process, which may be considered as deterministic events. And it is worth noting that information may be regarded somewhat as events.

In the process of tossing a true coin, there will be a continuous process as long as the coin is flipping. So, this is considered as kind of unknown, because there will be infinite and uncountable points and so an infinite and uncountable sample space. The continuous process is almost incontrollable, and it is in permanent change, and one can evaluate its limit rather than its exact value. So, there is less determination and more approximation. Moreover, science cannot visualize the discrete case precisely, except through its continuous case. So, the pervious paper (discrete case) has given some description of the continuous case, see corollary V (Jughaiman A., 2023). Now, suppose that if one monitors a coin at its flipping at specific points and specific time intervals, to get the result of the head and tail such that $(H,T):(\frac{49}{100},\frac{51}{100})$. How this result, could be repeated? So, if this question is necessary, then the answer is also necessary. The notion of continuity dates back to Leonhard Euler (1707-1783). But the more modern version of continuity is credited to Bolzano (1817) and Cauchy (1821). And, both Bolzano and Cauchy were concerned with continuity on an interval, rather than continuity at a point (Stoll, 1997).

In reality, the epistemological value of the theory of probability is revealed only by limit theorems. Moreover, without limit theorems it is impossible to understand the real content of the primary concept of probability. Historically, there are five considerable limit theorems, four of them deal with a sequence of independent events, (Gnedenko & Kolmogorov, 1968).

Methods

Experiment purpose: This experiment consists of observing the appearance of probability fractions (interval) of head or probability fractions (interval) of tail, in continuous processes. And to answer the following question: how does the continuous probability value behave?

Experiment steps: This paper will use a coin, but after shaping it into a spherical shape (true ball), with radius of r. Then the ball will be divided theoretically into n ordered closed circles with the replacement, for every half separately, where n is unkown. Also, consider that the peak of each upper half of the ball is success, figure 1.

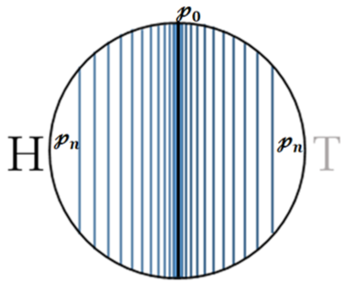

Figure 1. Every half is divided into infinite uncountable fractions. $\wp_n$approachs 1, while $\wp_0$ will represent $\emptyset$.

It should be noting that, in the descrete case, i is used to denote the number of fraction that is passable to happen in every throwing separately, (Jughaiman A., 2023). While in the continuous case, this present paper uses an interval to record fractions in some arbitrary time intervals. And, on every point at the peak, supposes a cumulative value of probability fractions (circles) starting by zero in an ordered manner, taking $\wp_0 = 0$ as a lower fraction (minimum and lower bound) and taking $\wp_n = 1$, as an upper fraction (maximum and upper bound) in the closed interval $[0,1]$. But as the

points are infinite and uncountable, $\wp_i$ will be unknown. Where $\wp_i$ is approaching $\pm\infty$. Here, if $\wp_0$ and $\wp_n$ are expressed as points, also they are expressed as intervals $\wp_{i=[0,0]}$, $\wp_{i=[1,1]}$ or into an interval $\wp_{i=[0,1]}$.
In this experiment there are two cases; if the ball moves in a straight line, then there is one direction. The second, if the ball moves on unlevel surface, then there are different directions. In addition, this present paper will use the following procedures and derived expressions, figure 2.

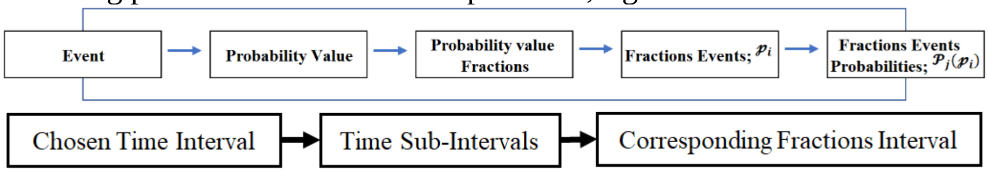

Figure 2. Used procedures and derived expressions

Also, in this present paper, interval is mean the fractions interval, unless determined it. And, for the time sub-interval (a, b), this paper considers that, the longest one to be the requiared time that the ball spends, through a distance from zero to $\frac{\pi}{2}$, in its fastest move. And the shortest one is when the ball still without any movement.

Results

If two arbitrary points are selected on the surface of a sphere of radius r. Then, the probability that an arc of a great circle passing through these points to make an angle less than α, where $\alpha < \pi$, is the area of the half surface of sphere minus this area multiple by cosine α, then all that divided by the whole area of the sphere surface. So, $p = \frac{2\pi r^2 - 2\pi r^2 \cos\,\alpha}{4\pi r^2} = \frac{2\pi r^2(1-\cos\,\alpha)}{4\pi r^2} = \frac{(1-\cos\,\alpha)}{2} = sin^2\frac{\alpha}{2}$, figure 3. See problems in (Sveshnikov, 1968).

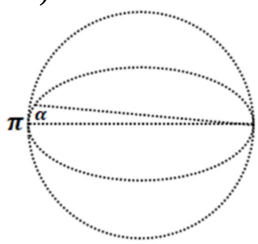

Figure 3

The probability that an arc of a great circle passing through points to make an angle equal to π, is the area of the half surface of sphere, $p = \frac{2\pi r^2}{4\pi r^2} = \frac{1}{2}$, which equals to $p = \frac{p(H)}{S} = \frac{1}{2}$.

Case 1: One direction

1. If the ball moves on a straight line, starting from zero with a constant velocity. Consider the following: every sub-interval of time will be corresponding to the fractions' interval i in movement of rotation of 90°. Here, the sample space will be all points on the great circle, figure 4.

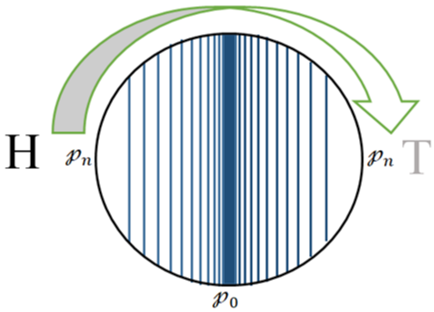

Figure 4. *H* is appearing completely when $\wp_0$ coincides with the x-axis 180°clockwise

Also, there will be an ordered head and tail fractions, but no randomness. Nevertheless, if one applies the probability theory, then the outcomes for $\{(H),(T)\}$ will be the ordered pairs: $\{(0,1),(0,1),\ldots+\infty\}$. Which are interpreting, why the probability value takes the values of 1 and 0, and why the area under the density curve should be equal 1 in the case of continuous random variable.

2. If the ball moves on a straight line, starting from zero with variant velocity then, there is only one direction for some intervals; $[0,t_1],[0,t_2],[0,t_3],\ldots,[0,t]$. Such that the values of $\wp_i$ tend to $\wp_n$ then tend to $\wp_0$ again, and so on. And, as the rotation degree will be random for unconstant velocity, so the corresponding fractions on the curve will be unequal for equal time sub-intervals, figure 5.

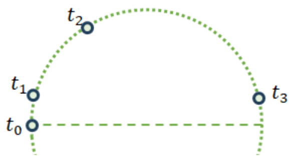

Figure 5. Equal and constant time intervals with unequal fractions intervals

Also, if the speed of ball increases arbitrarily, then the intervals become narrower and narrower. And this can draw the normal distribution form (Gaussian distribution form), figure 6. Where, the probability distribution of a sum of independent random variables tends to become gaussian as the number of random varibles being summed increases without limit. Such as, the shot noise generated in a thermionic vacuum tube, and the voltage fluctuation produced by thermal agitation of electrons in a resistor

(Davenport, 1958). And as the ball stops, then the continuous case will be breakup at a discontinuity point and called the discrete case.

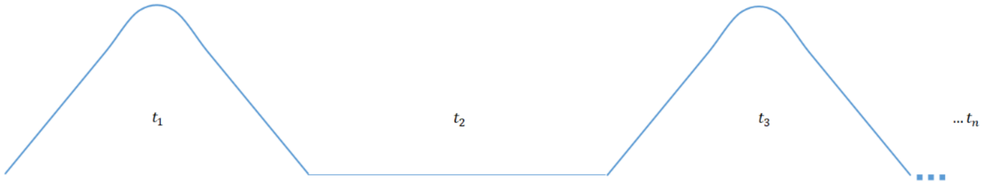

Figure 6

Proposition I: For every random process, the number of trial of any process is at least equal to the square number of the real events, m^2.
As this proposition may seem trifle, it is also useful. And it may reflect one of the sampling techniques as well as counting techniques.
Proof: In principle, the number of trials in the case of a true coin, in a discrete process should be at least four trials to get head or tail, table 1.

Table 1. The chances of every event $\boldsymbol{m^2}$, not the permutation $m!$

H	T	1	2	3	1	2	3	4	⋯
T	H	2	3	1	2	3	4	1	⋯
		3	1	2	3	4	1	2	⋯
					4	1	2	3	⋯
					⋮	⋮	⋮	⋮	⋱

Because in the case of throwing a ball, if one records the fraction that the ball movement through the interval ended at, then at every instant of time there will be a new fraction. Such that, for infinite uncountable events (points) $\wp_i$ in the closed interval $[0,1]$, then $\sum_{i=0}^{\infty} P(\wp_i) = 1$ and if this quantity is less than 1, then there are missing events. In other words, there are no sure or impossible events. And, as this is impossible because of the uncountable aspect of the real numbers. There is always need to enough intervals that should reflect the main characteristics in any experiment, besides avoiding the errors. Also, while this could be use to prove the uncountable aspect of the real numbers, continuity helps, on the other hand, to reduce the errors that cannot be recognized in the discrete process.

In addition, in the case of discrete processes the outcome of tossing coin or throwing ball, has values with large difference, such as tossing a true coin twice with outcomes of 0 and 1. While in the continuous case, there is a value at every instant, and this value is very close to the previous value or the next value. So, the function should be a continuous function. “A small change in x produces only a small change in the function value $f(x)$. This is not accurate description, but rather device to help develop an intuitive feeling for continuous functions”, (Swokowski, 1988).

For a sequence $\{A_n\}, n = 1, 2, \dots$, the set of all point which belong to almost all A_n (all but any finite number) is called the inferior limit of A_n and

$\liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$. Also, the set of all point which belong to infinitely many A_n is called the superior limit of A_n and $\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. So, $\liminf A_n \subset \limsup A_n$, but if $\liminf A_n$ and $\limsup A_n$ are equal to the same set A, then A is called the limit of A_n. Also, a sequence $\{A_n\}, n = 1, 2, ...$, is said to be monotone if it is either nondecreasing $A_1 \subset A_2 \subset \cdots$, and to write $A_n \uparrow$, or if it is nonincreasing $A_1 \supset A_2 \supset \cdots$, and to write $A_n \downarrow$. So, every monotone sequence is convergent and $\lim A_n = \bigcup A_n \; or \; \bigcap A_n$ according as $A_n \uparrow$ or $A_n \downarrow$ (Loève, 1977).

Corollary I: For a continuous process at some intervals, the probability of fraction events can take negative probability values in sub-sequence of a sequence that sum to zero:
$\mathcal{P}_{j\downarrow}(\wp_i)$, where $j = 1, \dots, m$ and $i = 1,2, \dots, \infty$.
Proof:

From probability axioms,
$0 \le \wp_i \le 1$, for $i = 1,2, \dots \infty$.
And if $\wp_i$, $i = 1,2, \dots \infty$ are mutually disjoint sets in $\mathfrak{F}$ then,
$\bigcup_i^{\infty} \wp_i = \sum_i^{\infty} \wp_i$.

Also, as $\mathcal{P}_H(\wp_{i=0,1,..n}) + \mathcal{P}_T(\wp_{i=0,1,..n}) = 1$ and,
$\mathcal{P}_H(\wp_{i=0,1,..n}) = \mathcal{P}_T(\wp_{i=0,1,..n})$ then $\mathcal{P}_H(\wp_{i=0,1,..n}) = \frac{1}{2}$.

And, if the ball moves through an interval i, where is $\frac{\pi}{2} < i < \pi$ and if,
$\sum_{0 \le i \le \frac{\pi}{2}}^{\infty} \wp_i = \sum_{\frac{\pi}{2} \le i \le \pi}^{\infty} \wp_i = \frac{1}{2}$. Then, $\sum_{0 \le i \le \frac{\pi}{2}}^{\infty} \wp_i + \sum_{\frac{\pi}{2} < i < \pi}^{\infty} \wp_i > \frac{1}{2}$, but this is contrary to the probability axioms. Therefore, one of the two series must be negative. So,
$0 \le \sum_{0 \le i \le \frac{\pi}{2}}^{\infty} \wp_i + \sum_{\frac{\pi}{2} < i < \pi}^{\infty} \wp_i \le \frac{1}{2}$ multiple by -1, gives
$-\frac{1}{2} \le -\sum_{0 \le i \le \frac{\pi}{2}}^{\infty} \wp_i - \sum_{\frac{\pi}{2} < i < \pi}^{\infty} \wp_i \le 0$, then take $\sum_{0 \le i \le \frac{\pi}{2}}^{\infty} \wp_i = \frac{1}{2}$ and adding $\sum_{0 \le i \le \frac{\pi}{2}}^{\infty} \wp_i$, then $0 \le -\sum_{\frac{\pi}{2} < i < \pi}^{\infty} \wp_i \le \frac{1}{2}$, and $-\frac{1}{2} \le \sum_{\frac{\pi}{2} < i < \pi}^{\infty} \wp_i \le 0$.

As a result, the negative value here is necessary, but in this case the sum of the probability values in an infinite interval for both, head and tail, will be zero.

Now, suppose that the ball moves in one direction at constant speed through the time interval $[t_0, t]$ which is corresponding to the fraction interval $[0, \frac{\pi}{2}]$, then the displacement $\theta = v\Delta t$, and if $v = 1$ then, $\theta = \Delta t = t$.
Let $t = n$, then $n - 1 \le t \le n + 1$. And,
$n \le t + 1 \le n + 2$. So, $t + 1 \le n + 2$ or $n \le t + 1$.

Also, if $\mathcal{P}_H(\wp_{i=0,1,..n}) = \mathcal{P}_{H\uparrow}(\wp_{i=0,1,..n}) + \mathcal{P}_{H\downarrow}(\wp_{i=1,..n})$. And, if the integration of $[\int_0^{\frac{\pi}{2}} f(x)\,dx] + [\int_{\frac{\pi}{2}}^{0} f(x)\,dx] = 0$.

Recall $\mathcal{P}_J(\wp_i)$ in the discrete case, where $\sum_j^m \sum_i^n \mathcal{P}_J(\wp_i)_j = \sum_j^m \sum_i^n \frac{1}{n+1}\left[1 - \frac{J(n-i)}{Jn}\right]$, and for r rounds, this will be; $(-1)^{r+1}\sum_i^n \mathcal{P}_{H\uparrow}(\wp_i) + (-1)^{r+1}\sum_i^n \mathcal{P}_{H\downarrow}(\wp_i) + \cdots$ for $r = 1,2,\ldots$

$$\sum_i^n \mathcal{P}_{H\uparrow}(\wp_i) = \mathcal{P}_{H\uparrow}(t_0,t) = \frac{(-1)^2}{t}\int_{t_0}^{t} \mathcal{P}_{H\uparrow}(\theta)d\theta \ = \frac{1}{t}\int_{t_0}^{t} 1 - \frac{(t-(t-t_0))}{t}\ d\theta.$$

$$= \frac{1}{t}\int_{t_0}^{t} \frac{t-t+(t-t_0)}{t}\ d\theta.$$

$$= \frac{1}{t}\int_{t_0}^{t} \frac{(t-t_0)}{t}\ d\theta = \frac{1}{t^2}\int_{t_0}^{t} \theta\ d\theta = \frac{1}{t^2}\left[\frac{\theta^2}{2}\right]_{t_0}^{t} = \frac{t^2}{2t^2} = \frac{1}{2}\,Result\ 1.$$

Also, for the time interval (t, t_0).

$$\sum_i^n \mathcal{P}_{H\downarrow}(\wp_i) = \mathcal{P}_{H\downarrow}(t,t_0) = \frac{(-1)^3}{t^2}\int_{t}^{t_0}(t_0 - t)\,d\theta.$$

$$= \frac{-1}{t^2}\int_{t}^{2t} \theta\,d\theta = \frac{-1}{t^2}\left[\frac{\theta^2}{2}\right]_{t}^{2t} = -\frac{t^2}{2t^2} = -\frac{1}{2}\,Result\ 2.$$

And, $\mathcal{P}_H(\wp_{i=0,1,..n}) = \mathcal{P}_{H\uparrow}(\wp_{i=0,1,..n}) + \mathcal{P}_{H\downarrow}(\wp_{i=1,..n}) = 0\ Result\ 3$.

Here, for all the following intervals, $(0,0),(0,\pi),(\,0,2\pi),(\,0,3\pi),\ldots,(0,\ n\pi)$ the probability value will be 0. On the other hand, these results reflect the fact that $\wp_{(+\infty)} = 1$, and the fact that $\wp_{(-\infty)} = 0$. Also, these resultes reflect that the nil value does not only mean that the impossible event is happening, but it also means that there is no discrete process. At the same time, in the discrete case, nil value does not only mean that the impossible event is happening, but it also means that there is a continuous process. Also, this value may provide an explanation of the limit value for $\lim_{n\to\infty}\frac{1}{2^n} = 0$.

Also, in $Result$ 3, if $\mathcal{P}_{H\uparrow}(\wp_{i=0,1,..n})$ is multiplied by 2, as a weight, then it will be; $\mathcal{P}_H(\wp_{i=0,1,..n}) = 2\mathcal{P}_{H\uparrow}(\wp_{i=0,1,..n}) + \mathcal{P}_{H\downarrow}(\wp_{i=1,..n}) = \frac{1}{2}$, $Result\ 3'$. Also, this result could be obtaind by sequaring the both functions. But for the distribution function and under the probability axioms, the case is still different: "A function of $t, F(t)$ is never decreasing", (Uspensky, 1937). So, to not loss generality in results 1, 2 and 3, let $t + 1 \geq n$, where θ should be $0 \leq \theta \leq t$, and if $t = 1$ then the integration is;

$\sum_i^n \mathcal{P}_{H\uparrow}(\wp_i) = \sum_i^n \mathcal{P}_{H\downarrow}(\wp_i) = \mathcal{P}_{H\uparrow}(t_0,t) = \frac{1}{t+1}\int_{t_0}^{t} \frac{t-t+(t-t_0)}{t}\ d\theta$. Then

$$\frac{1}{t(t+1)}\int_{t_0}^{t} t - t + (t - t_0)\ d\theta = \frac{1}{t(t+1)}\int_{t_0}^{t} \theta\ d\theta = \frac{1}{t(t+1)}\left[\frac{\theta^2}{2}\right]_{t_0}^{t} = \frac{1}{4}.\ Result\ 4.$$

And, this explains that the ball is divided by 4 quarters, every face has two attached quarters with positive signs. Also, if one supposes that the value of t (displacement) is increasing infinitely such that $0 \leq \theta < +\infty$, then the probability of probability value is also increasing to approach the value of $\frac{1}{2}$, but it does not exceed it. Which proves numerically that the limit of the function is $\frac{1}{2}$, table 1. At the same time, as the value of t decreases infinitely, then the probability of probability value is decreases to approaches zero, but it does not exceed it. So, these also explain the results of 1, 2 and 3.

Table 2. A small change in t produces a small change in the function. Also, whatever the acceleration of t is, the function has less acceleration

t	t^2	$\frac{t^2}{2t^2+2t}$
1	1	0.25
2	4	0.333333333
⋮	⋮	⋮
1,000,000,000	1E+18	0.5

Also, in continuous processes the conditional concept could be considered. So, for some intervals;

$\mathcal{P}_{H\uparrow}(\wp_{i=\frac{n+i}{2},\dots,n} \mid \wp_{i=1,\dots,\frac{n}{2}}) = \mathcal{P}_{H\downarrow}(\wp_{i=\frac{n-i}{2},\dots,1} \mid \wp_{i=n,\dots,\frac{n}{2}})$.

In this case $\wp_{i+1\uparrow}$ occurring if $\wp_{i\uparrow}$ occurs is given, and this is a result of the physical situation that subjected to the conditionality. And it is sufficient, that every point in $\wp_{i\uparrow}$ is also in $\wp_{i+1\uparrow}$ which is satisfying that the points is in $\wp_{i\uparrow}\wp_{i+1\uparrow}$.

For a probability space $(\mathfrak{F}, \mathfrak{B}, \mathcal{P}(\cdot))$ if $\mathfrak{F}$ is uncountable, then $\mathfrak{B}$ cannot in general be the set of all subsets. But, once one has one probability defined over $(\mathfrak{F}, \mathfrak{B})$, then one can define other probabilities that are called conditional probabilities (Tucker, 1967).

Corollary II (From Multiplication Rule): For every n events $\wp_1, \dots, \wp_n$ for which $\mathcal{P}_{j=1,\dots,m}(\wp_1 \dots \wp_{n-1}) > 0$. Then,

$\mathcal{P}_{j=1,\dots,m}(\wp_1 \dots \wp_n) =$
$\mathcal{P}_{j=1,\dots,m}(\wp_1)\mathcal{P}_{j=1,\dots,m}(\wp_2|\wp_1) \cdots \mathcal{P}_{j=1,\dots,m}(\wp_n|\wp_1 \dots \wp_{n-1})$.

For the theorem and proof see (Tucker, 1967).

Corollary III (From Theorem of Total Probabilities): If $\mathcal{P}_{j=1,\dots,m}(\bigcup_{i=1}^{n} \wp_i) = 1$, where $\{\wp_i\}$ are a finite or denumerable sequence of disjoint events, if $\mathcal{P}_{j=1,\dots,m}(\wp_i) > 0$ for every i, and if $A \in \mathfrak{B}$. Then, $\mathcal{P}_{j=1,\dots,m}(A) = \sum_{i=1}^{n} \mathcal{P}_{j=1,\dots,m}(A|\wp_i)\,\mathcal{P}_{j=1,\dots,m}(\wp_i)$. For the theorem and proof see (Tucker, 1967). It should be noting that, in the present case $\mathcal{P}_{j=1,\dots,m}(\wp_0) \geq 0$, so it is excluded.

But the difference in the discrete case is that the events are independent in every throwing, where in the present case every event is supposed to be dependent on the previous event. “Among Markov’s own significant contributions to probability theory were his pioneering investigations of limit theorems for sum of dependent random variables and the creation of a new branch of probability theory, the theory of dependent random variables that form what we now call a Markov chain". (Shiryaev, 2016).

3. If the ball moves in one direction but it diffuses in all directions at a speed that is more than the speed of its movement, then the points turn away. And at every instant there would be unknown point (uncountable), figure 7.

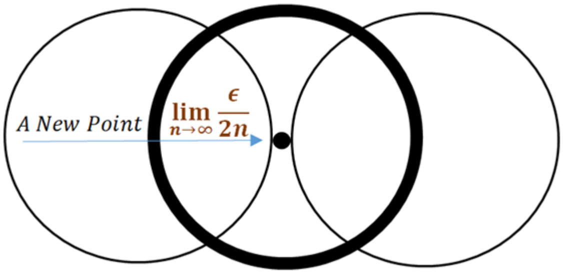

Figure 7. As point is undefined, it is also unknown (uncountable)

Case 2: Different directions

The sample space will be all points on the ball. If one considers that all events are equally likely, then it will also bring result 3, figure 8.

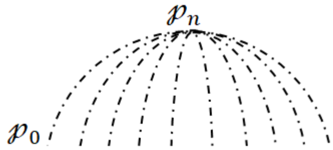

Figure 8

Definition I: Let Ω be a measurable subset of n-dimensional Euclidean space with positive, finite Lebesgue measure. Let further $\mathcal{A}$ be the set of all measurable subsets of Ω and $\mu(A)$ the n-dimensional Lebesgu measure of the measurable set A. Let $P(A)$ be defined by $\frac{\mu(A)}{\mu(\Omega)}$, and if $(\Omega, \mathcal{A}, P)$ is a kolmogorov probability space. Then in this probability space probabilities may be obtained by geometric determination of measures (Rényi, 1970).

The selections of measurable sets and of concepts of limit in range-spaces are rooted in the properties of the Euclidean line: Real line $R = (-\infty, +\infty)$ with Euclidean distance $|x - y|$ of points (numbers, reals) x, y. Species of spaces vary according to the preserved amount of these properties, an amount which increases as we pass from separated spaces to metric spaces, then to Banach spaces and to Hilbert spaces (Loève, 1977).

Theoritacly, if one bends the portion of the real line of the closed interval [0,1] as figure 9 shows. Then, this will represent the infinite uncountable fractions (circles).

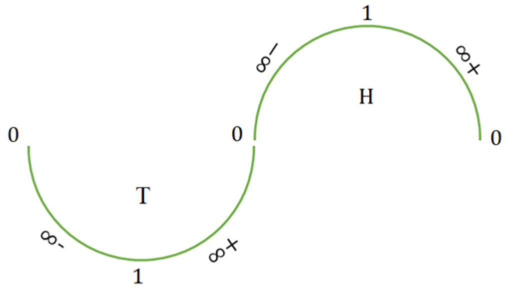

Figure 9. The portion of the real line that bends over every quarter of the ball

It is well known that we can establish a one-to-one correspondence between all real numbers and all points on the line. Also, a correspondence may be established between all pairs of real numbers (x_1, x_2) and all points in a plane or between all triplets of real numbers (x_1, x_2, x_3) and all points in a three-dimensional space. Moreover, the length of a finite interval (a, b) in $R_1$is the non-negative quantity and zero for the degenerate interval. Thus, with every interval $i = (a, b)$ associated a definite non-negative length, which may be finite or infinite. And, we may expressed this by saying that the length $L(i)$ is non-negative function of the interval i and writing, $L(i) = b - a$, or $L(i) = +\infty$, (Cramér, H., 1946). In the present case, interval $\wp_{i=[0,0]}$ is a degenerate interval. In addition, in the problem of uncountable points, if area is used instead of numbering, for the process of tossing a true coin then the probability of head will be the area of head / the area of the coin. Intuitively, if the ball considered as one event that divided by many events (portions), then the entire area cannot be divided by unequal portions. Also, in this present case for each face, if the two intervals $[0, 1]$ and $(0, 1)$ are considered separately. So, for some successive intervals without common points, the whole fractions sub-intervals will take a length of less than or equal to 1. So, it is possible to represent the function $L(i)$ by the function $\mathcal{P}_j(\wp_i)$.

Also, if the end points of any interval will be from an infinite uncountable sample space, then the intervals are infinite and uncountable intervals.

Geometrcaly, by a random vector in R^3 is meant a vector drawn in a random direction with a length L which is a random variable independent of its direction. The probabilistic properties of a random vector are completely determiend by those of its projection on the x-axis, and using the latter it is frequently possible to avoid analysis in three dimensions, (Feller, 1971).

As the ball moves in different directions, with various velocities then there would be vectors and scalars. And, every vector has initial point v_0, terminal point v_1 and length as a magnitude, which will be the time interval length. Which also depends on the velocity. Simply, every point a has a probability value equal to $\frac{a}{t}$.

On the other hand, in the case where the ball is still without movement, every point at a random position on the ball could sketch a ball, and as the points are uncountable, there will be uncountable balls. Also, when the ball moves, then every point takes the position of the other one. So, if there are uncountable balls, then the probability to pull one of them, will be such as the experiment of drwing a random ball in a continuous process. And for arbitrary time interval, it is impossible to find how many balls can be drawen.

Nevertheless, since there are random velocities and random directions, then the value will behave as following:

1. The points at the peak remain around its position, if the ball move around its axle , so the value remains within a bounded range.
2. The points at the peak remain around its orbit, if the ball move around this orbit, so the value remains within the some greate circles.
3. The point move in random directions, so the value changes randomly.

From these, if one supposes that every point is moving on random orbits, then by use these orbits instead of points, there are fixed infinite uncountable orbits. And the argument being, what is the probability value of every orbit the moving ball may take. And, how long the interval is for the ball in its moving, remains at a specific half (head or tail). Therefore, there would be some kinds of behaviors such that:

1. The behavior of the value in a specific time interval.
   It will subjected to the conditions of the existing case. Meant, the events occurrence will not be equally likely (limited-time interval).
2. The behavior of the value in an infinite time interval.
   It will reflect the value behavior that depends on the ball movement behavior. Meant, the events occurrence will be equally likely.

Approach

To approach the ball movement problem to the stochastic processes' problems. Consider the following examples:

"In the Bohr model of the hydrogen atom, the electron may be found in one of certain admissible orbits. This is a Markov chain with an infinite number of states (although in principle only)", (Gnedenko, 1963).

Also, for example: "A man starts at the origin and takes a step in any direction of length Δ. He then stops, selects a new arbitrary direction, and proceeds to take another step of length Δ in this new direction. He continues his walk for n steps. The angles through which he proceeds on the n steps are chosen independently and at random and thus may be taken as independent random varibles. Therefore, let a_k be a random variable whose value determines the angle made with the x axis in the k^{th} step and assume that it is uniformly distributed from 0 to 2π, that is, the frequency function is $\frac{1}{2\pi}$ for $0 \leq a_k \leq 2\pi$ and zero elsewhere." Also, "for a stationary and ergodic random process, we consider the motion of a perfectly elastic billiard ball on a frictionless circular table with perfectly elastic boundaries. We assume the diameter of the ball to be zero and its speed to be a constant v. It is clear that upon each impact with the boundary of the table the direction of motion of the ball changes by the fixed amount α. From elementary geometry it is found that all paths are tangent to an inner circle of radius $r_0 = R\cos\left(\frac{\alpha}{2}\right)$ and that each chord length is $2R\sin\left(\frac{\alpha}{2}\right)$", (Laning, et., 1956).

Discussion

As soon as making partions on the ball such as if considering alternative design, figure 10 to the right, then there would be discontinuity points, for the true coin state from H to T. Also, figure 10 at the middle, consider finite sample space for the structure of single-member event, which is an event that contains exactly one description, (Parzen, 1960). In this case, the exactly description will be positive finite interval or negative finite interval, and the sum of the fractions of probability will be $\wp_n$ or 1, for every face, and $\frac{1}{2}$ for the composite function respectively, (Jughaiman A., 2023). In addition, it is possible to record the negative values (probability negative values) on the quarter of every half, by considering that the quarters of every half are equally likely, no matter what the directions are. At the same time, the probability of probability will be positive.

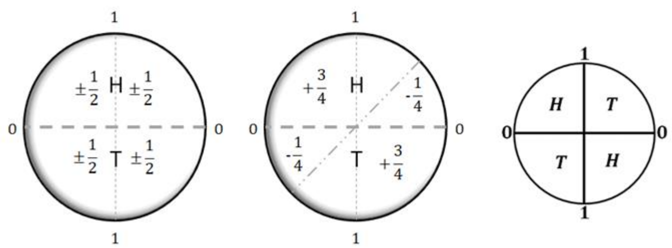

Figure 10. To left, the sum of possibility for any face is $\pm\frac{1}{2}$. And to right it cannot in a continuous process to flip the coin from face to face directly, or without zero fraction, $\emptyset$.

In $result\ 4$, for any interval less or more than $t = 1$, such as (t_i, t_{i+r}), and because the two quarters of every half are symmetric, also because the function, $\mathcal{P}_{H\uparrow}(t_0, t) = \frac{1}{t+1}\int_{t_0}^{t} \frac{t-t+(t-t_0)}{t}\, d\theta$, is a cumulative function, it should be multiplied by v^{-1} to be; $\mathcal{P}_{H\uparrow}(t_i, t_{i+r}) = \frac{v^{-1}}{t+1}\int_{t_i}^{t_{i+r}} \frac{t-t+(t_{i+r}-t_i)}{t}\, d\theta$. This is to give a probability value that equals to the fraction area frequency. Also, every quarter of every face could be consider as a mirror for the other quarter.

On the other hand, for unequal fractions intervals, there would be unequal areas. So, one cannot assign an event to every subset of the area, where areas cannot be defined for every subset such that it is completely additive and the areas of cogruent figures are equal, in general, the distribution of probability is said to be uniform, if the probability that an object situated at random lies in a subset can be obtained according to the definition (I) form a geometric measure invariant under displacement (Rényi, 1970). Also, in the continuous process, the occurrence of H can not be discontinuity point of the occurrence of T, they are considered as one phenomenon, which often being as one homogenous physical body.

To accommodate and to admit the negativity, in $results\ 1, 2$ and 3, consider the following: First of all, probability theory has treated the non-decreasing and non-increasing functions to find the density functions precisely. “The case in which the function $g(\cdot)$ is differentiable at every real number x and, further, either $g'(x) > 0$ for all x or $g'(x) < 0$ for all x”, (Parzen, 1960). But in principle in the present case, note that all fractions are positive, and when the probability value behaves in non-increasing sequance, this sequance should be a sub-sequance of a sequance which sums to value that is satisfying the probability axioms which is the zero value, as a particular case. From here, if the probability value is admited to take negative sign, it takes it to reflect the continuous process in some directions. On the other hand, it could be explained such that: the values from 0 to 1 being in the success intervals, for the occurrence of fractions that precede or tend to the sure event. At the same time, the value from 1 (0) to 0 (-1) being in the failure intervals, for the occurrence of fractions that precede or tend to the impossible event. For example, in one direction movement, if the interval $(-\frac{1}{2}, -1)$ means the actual occurrence of an event, then the next interval will be the complement event. “There are at least two directions from which the fundamental axioms of probability theory may be approached: probability theory concepts may be built up, mechanically, from the concept of the event and its probability, or they may be derived as special applications of the theory of measurable spaces” (Allen, 1976).

How to prove that $\wp_{i=0,1,..n}$ has an infinite and uncountable number of points? This can be proved from the fact that the real numbers are uncountable which proveded in 1873 by Cantor in diagonal argument from that the closed interval $[0,1]$ is uncountable, (Stoll, 1997).

Also, for infinite time sub-intervals, there would be infinite fractions sub-intervals which are necessarily with common points. Moreover, for a random velocity, these sub-intervals have random lengths, which are considered as random variables, where the endpoints could also be considered in specific fraction intervals as random variables, which are not necessarily independents. For more discussion, see (Justicz et al., 1990).

Conclusion

This paper concluded that, the experiment of tossing a true coin can be represented geometrically using the fractions and the composite probability function in a continuous case, which provides a sense that is visualizing the probability value behavior. Also, this paper concluded that the continuous process can explain clearly the theoretical concept of discrete process for probability values of 1 and 0. In addition, this paper proposes to include the counting principal in the experiment design.

This paper concluded that the sum of probability values in a move of rotation of 180° can represent the smallest picture of the continuous infinite rotation towards $\pm\infty$. Also, this paper concluded that the randomness in a continuous process is explained by the sample space and the recurrence (repeating) of event. While in the discrete process the randomness is explained by the extent of the difference of probability value that the event takes.

This paper concluded that the continuity could be considered as the nature of all events. Also, this paper concluded that the present experment can interpret the continutiy by the small change of fraction, compared to the large change of fraction in the discrete case. Also, this paper concluded that the negative values of probability do not appear in the discrete processes but they appear in the continuous processes. Therfore, the random processes could be described in more details in the continuous processes.

This paper concluded that, there are maxima and minima limits for the given function which reflect the probability value behavior. Also, it is clear that the probability of non-decreasing sequence is equal to the probability of the non-increasing sequence, which are cancelling each other. On the other hand, it is possible to assuming that the non-increasing values, take the meaning of failure for the event under study, then the meaning of success for the complement event, that for a given purpose.

This paper concluded that, the negative values at some points in some intervals is considered as a particular case (trial), and they can not be discontinuity poins from the continuous case as negative values, but as positive discrete values. Also, the adopting of the negative probability value is depending on the extent of the explanation of the non-increasing sequence, and on the extent of the need to negative value. On the other hand, by some amendments of the experiment design, the negative values could be calculated by the composite function to give positive values with $\mathcal{P}_H(\wp_{i=0,1,..n}) + \mathcal{P}_T(\wp_{i=0,1,..n}) = 1$. So, the probability of probability has a mathematical necessity. Also, the composite function can be used in the conditional probability, if $\mathfrak{F}$ is uncountable.

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