A HAMILTON-JACOBI TREATMENT OF DISSIPATIVE SYSTEMS

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Abstract

Dissipative systems are investigated within the framework of the Hamilton-Jacobi equation. The principal function is determined using the method of separation of variables. The equation of motion can then be readily obtained. Three examples are given to illustrate our formalism: the damped harmonic oscillator, a system with a variable mass, and a charged particle in a magnetic field.

Keywords: Hamilton-Jacobi Equation, Dissipative Systems

1. Introduction

It is well known that the energy concept is almost indispensable in the analysis of physical systems. Such systems can be studied in terms of their sources and sinks of energy. A dissipative system is that which cannot store all the energy imparted to it by an external source, losing energy through some sink (Greiner, 1953).

In this work, dissipative systems are investigated using the Hamilton-Jacobi equation (HJE). This equation is simplified using the separation-of-variables technique. The corresponding principal function S is found. The equation of motion can then be derived from this function, which represents the energy of the system in terms of the generalized coordinates and momenta. This, in turn, is used as a basis for theso-called canonical quantization using the WKB approximation, thereby obtaining the corresponding Hamiltonian and Schrödinger's equation (Das, 2005).

In the Hamiltonian formulation of non-conservative systems, several methods have been devised to include dissipative effects. The earliest method invokes the so-called Rayleigh dissipation function, which is valid when the frictional forces are proportional to the velocity. However, in this method, another scalar function is needed, in addition to the Lagrangian, to specify the equations of motion. This function does not appear in the Hamiltonian; so it is of no use when attempting to quantize frictional systems. Another method, developed by Bateman (Bateman, 1931), introduces auxiliary coordinates in the Lagrangian that describe a reverse-time system with negative friction. This method leads to extraneous solutions, and the physical meaning of the momenta is not clear. Further, Bauer (Bauer, 1931) proved that it is impossible to use a variational principle to derive a single linear dissipative equation of motion with constant coefficients.

Recently, a completely different approach -the canonical- has been developed for investigating singular systems(Rabei, 1992).A general method for solving the Hamilton-Jacobi partial differential equation(HJPDE) for constrained systems has been proposed (Nawafleh, 2004) and (Rabei, 2002). The present work extends this framework, for the first time, to dissipative Lagrangian systems.

The first step here, then, is to construct HJPDE for dissipative systems (Section2). Three systems are examined within this framework (Section3): the damped harmonic oscillator(together with the RLC circuit and a viscous liquid); a system with a variable mass; and a charged particle in a magnetic field.

2. Hamilton-Jacobi Formalism

We start with the Lagrangian

$$L = L_0(q,\dot{q})\, e^{\lambda t}. \tag{1}$$

Here $L_0(q,\dot{q})$ stands for the Lagrangian of the corresponding conservative system; it represents the system's ‘physical’ Lagrangian, which means the kinetic energy minus the potential energy. The dissipation is incorporated through λ, which is a damping factor (λ>0). As usual, the generalized momentum is defined by (Fowles, 1993).

$$p = \frac{\partial L}{\partial \dot{q}}.$$

The corresponding Hamiltonian is

$$H_0 = p_i \dot{q}_i - L. \tag{2}$$

HJE for dissipative systems is a first-order, non-linear partial differential equation, the kinetic energy being, in general, a quadratic function of momentum of the form

$$H_0(q_1, q_2, ..., q_N; \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, ..., \frac{\partial S}{\partial q_N}; t) + \frac{\partial S}{\partial t} = 0.$$

The generalized momenta do not appear in this equation, except as derivatives of Hamilton's principal function S, which is a function of the N generalized coordinates $q_1, q_2, ..., q_N$ and the time t.

Now, if S $(q_1, q_2, ..., q_N; \alpha_1, \alpha_2, ..., \alpha_N)$ is a complete integral of HJE, the integrals of Hamilton's equations of motion will be given by

$$\frac{\partial S}{\partial \alpha_j} = \beta_j, \tag{3}$$

β_j being some constants.

This equation can be inverted to find the N generalized coordinates q_i as functions of α_j, β_j and t. The generalized momenta are

$$\frac{\partial S}{\partial q_j} = p_j. \tag{4}$$

Thus, the Hamilton-Jacobi function is given by

$$H_0(q, p) + \frac{\partial S(q,t)}{\partial t} = 0. \tag{5}$$

Since $L_0 \equiv T - V$ is the physical Lagrangian of the system, it follows that H_0 is the physical Hamiltonian representing the system's total energy: T+V (Goldstein, 1980).

The resulting action S is

$$S = \int e^{\lambda t} L_0 dt = \int (p_i \dot{q}_i - H_0) dt.$$

To build HJPDE, we must write S in the separable form

$$S(q, \alpha, t) = W(q, \alpha) + f(t), \tag{6}$$

where the time-independent function W(q,α) is the so-called Hamilton's characteristic function.

Differentiating Eq. (6) with respect to t, we find that

$$\frac{\partial S}{\partial t} = \frac{\partial f}{\partial t}. \tag{7}$$

From Eq.(5), it follows that

$$\frac{\partial f}{\partial t} = -H_0. \tag{8}$$

The left-hand side of this equation depends on t alone; whereas the right-hand side depends on q alone. Each side must then be equal to a constant independent of both q and t. Therefore, the time derivative $\partial S/\partial t$ in HJE must be a constant, usually denoted by (-α).

Thus,

$$S(q,\alpha,t) = W(q,\alpha) - \alpha t. \tag{9}$$

It follows that

$$\mathrm{H}_0\left(\mathrm{q}, \frac{\partial \mathrm{W(q)}}{\partial \mathrm{q}}\right) = \alpha.$$

3. Examples

3.1 Damped Harmonic Oscillator

The following Lagrangian is suitable for this system in one dimension (Bateman, 1931):

$$L(q,\dot{q},t) = \left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2\right) e^{\lambda t}, \tag{10}$$

m being the mass of the oscillator and ω the frequency of oscillation.
The linear momentum is given by

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}\ e^{\lambda t}. \tag{11}$$

This equation can readily be solved to give

$$\dot{q} = \frac{p}{m} e^{-\lambda t}. \tag{12}$$

The canonical Hamiltonian has the standard form

$$H_0 = p\dot{q} - L. \tag{13}$$

Substituting Eqs.(10),(11) and (12) into (13), we get the Hamiltonian

$$H_0 = \frac{p^2}{2m} e^{-\lambda t} + \frac{1}{2}m\omega^2 q^2\ e^{\lambda t}. \tag{14}$$

We shall now use a change of variables to solve this equation. Suppose that

$$y = q\, e^{\frac{\lambda t}{2}}. \quad (15)$$

Using the chain rule, we find that

$$p = \frac{\partial S}{\partial q} = \frac{\partial S}{\partial y}\frac{\partial y}{\partial q} = \frac{\partial S}{\partial y}\, e^{\frac{\lambda t}{2}}. \quad (16)$$

Then we have

$$p^2 = \left(\frac{\partial S}{\partial q}\right)^2 = \left(\frac{\partial S}{\partial y}\right)^2 e^{\lambda t} = p_y^2\, e^{\lambda t}. \quad (17)$$

Substituting Eqs.(17)and (15) into (14), we find

$$H_0 = \frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2 + \frac{1}{2}m\omega^2 y^2. \quad (18)$$

Then HJE takes the form

$$\frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2 + \frac{1}{2}m\omega^2 y^2 + \frac{\partial S}{\partial t} = 0. \quad (19)$$

This differential equation is the well-known Hamilton-Jacobi equation for the simple harmonic oscillator. Its solution is

$$y = \sqrt{\frac{2\alpha}{m\omega^2}}\sin\left((\beta + t)\omega\right). \quad (20)$$

In terms of q, using Eq.(15), we get

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\sin\left((\beta + t)\omega\right) e^{\frac{-\lambda t}{2}}. \quad (21)$$

In the limit $\lambda \to 0$, Eq. (21) is in agreement with the well-known result for the *free* harmonic oscillator, as it should.

One can follow the same steps outlined in this example to study other dissipative systems, such as the RLC circuit and a viscous liquid, as follows:

For the RLC circuit, an appropriate Lagrangian is

$$L(Q,\dot{Q},t) = \left(\frac{1}{2}L\dot{Q}^2 - \frac{Q^2}{2C}\right) e^{\lambda t}.$$

It follows that

$$H_0 = \frac{1}{2L}\left(\frac{\partial S}{\partial y}\right)^2 + \frac{y^2}{2C}.$$

The HJ function can be obtained as

$$S = \int \sqrt{2L\alpha - \frac{y^2 L}{C}}dy - \alpha t.$$

The resulting equation of motion is

$$Q = A\sin\left((\beta + t)\sqrt{\frac{1}{CL}} \right) e^{\frac{-\lambda t}{2}}.$$

For a viscous liquid in a tube, we have the following Lagrangian:

$$L(q, \dot{q}, t) = \left(\frac{1}{2} l\dot{q}^2 - gq^2 \right) e^{\lambda t},$$

where l is the length of the liquid column, g is the gravitational acceleration taken here as constant, and q represents the variations in the liquid height. Its Hamiltonian is given by

$$H_0 = \frac{1}{2l}\left(\frac{\partial S}{\partial y}\right)^2 + gy^2.$$

The HJ function can be obtained as

$$S = \int \sqrt{2l\alpha - 2gly^2}dy - \alpha t.$$

Finally, the equation of motion is

$$q = \sqrt{\frac{\alpha}{g}} \sin\left((\beta + t)\sqrt{\frac{2g}{l}} \right) e^{\frac{-\lambda t}{2}}.$$

3.2 System with a Variable Mass

A suitable Lagrangian for this system is (Razavy,2005):

$$L(q, \dot{q}, t) = \left(\frac{1}{2} m\dot{q}^2 - mgq \right). \tag{22}$$

Suppose that the mass changes with time according to

$$m = m_0 e^{\lambda t}.$$

Then

$$L(q,\dot{q},t)=\left(\frac{1}{2}m_0\dot{q}^2-m_0gq\right)e^{\lambda t}. \qquad (23)$$

Clearly, the damping factor here arises from the variation of the mass with time.

The linear momentum is given by

$$p=m_0\dot{q}\ e^{\lambda t}. \qquad (24)$$

The usual treatment gives

$$H_0=\frac{p^2}{2m_0}\ e^{-\lambda t}+m_0gq\ e^{\lambda t}; \qquad (25)$$

$$H_0+\frac{\partial S}{\partial t}=0.$$

Further, the principal function takes the form

$$S(q,t)=qN(t)+D(t). \qquad (26)$$

So one gets

$$\frac{\partial S}{\partial t}=qN'(t)+D'(t), \qquad (27)$$

and

$$\frac{\partial S}{\partial q}=N(t). \qquad (28)$$

With $p=\frac{\partial S}{\partial q}$, we have

$$p^2=\left(\frac{\partial S}{\partial q}\right)^2=\left(N(t)\right)^2. \qquad (29)$$

The corresponding HJE takes the form

$$\frac{1}{2m_0}\left(N(t)\right)^2\ e^{-\lambda t}+m_0gq\ e^{\lambda t}+qN'(t)+D'(t)=0. \qquad (30)$$

Matching powers of q, we get

$$\frac{-1}{2m_0}\left(N(t)\right)^2\ e^{-\lambda t}=D'(t), \qquad (31)$$

and

$$m_0 gq\, e^{\lambda t} + qN'(t) = 0. \tag{32}$$

After integration:

$$N(t) = N_0 - m_0 g \frac{e^{\lambda t}}{\lambda}, \tag{33}$$

and

$$D(t) = -m_0 g^2 \frac{e^{\lambda t}}{2\lambda^3} + N_0^2 \frac{e^{-\lambda t}}{2m_0\lambda} + \frac{gN_0 t}{\lambda} + D_0. \tag{34}$$

Putting Eqs.(33)and (34) into (26), we have

$$S = -m_0 gq \frac{e^{\lambda t}}{\lambda} + N_0 q$$

$$-m_0 g^2 \frac{e^{\lambda t}}{2\lambda^3} + N_0^2 \frac{e^{-\lambda t}}{2m_0\lambda} + \frac{gN_0 t}{\lambda} + D_0. \tag{35}$$

Thus,

$$\beta = \frac{\partial S}{\partial N_0} = q + N_0 \frac{e^{-\lambda t}}{m_0\lambda} + \frac{gt}{\lambda}. \tag{36}$$

That is,

$$q = \beta - N_0 \frac{e^{-\lambda t}}{m_0\lambda} - \frac{gt}{\lambda}, \tag{37}$$

and

$$p = \frac{\partial S}{\partial q} = N_0 - m_0 g \frac{e^{\lambda t}}{\lambda}. \tag{38}$$

3.3 A Charged Particle in a Magnetic Field

As a final example, let us consider the motion in two dimensions of a charged particle under the influence of a central force potential, $V=kr^2/2$, as well as an external constant magnetic field perpendicular to the plane of motion: $\vec{B} = B_0\hat{k}$.

The vector potential is

$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{r} = \frac{1}{2}B_0(-y\hat{i} + x\hat{j}).$$

The Lagrangian is [9]

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{q}{c}(\vec{v}.\vec{A}) - \frac{k}{2}(x^2 + y^2).$$

In the presence of damping effects, the Lagrangian becomes

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\, e^{\lambda t} + \frac{q}{c}(\vec{v}.\vec{A})\, e^{\lambda t} - \frac{k}{2}(x^2 + y^2)\, e^{\lambda t}. \qquad (39)$$

With $\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$,

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\, e^{\lambda t} + \frac{qB_0}{2c}(x\dot{y} - y\dot{x})\, e^{\lambda t} - \frac{k}{2}(x^2 + y^2)\, e^{\lambda t}. \qquad (40)$$

To simplify, plane polar coordinates are used:

$x = r\cos\theta;$

$y = r\sin\theta.$

Then Eq.(40) becomes

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)\, e^{\lambda t} + \frac{qB_0}{2c}r^2\dot{\theta}\, e^{\lambda t} - \frac{k}{2}r^2\, e^{\lambda t}.$$

The conjugate momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}\, e^{\lambda t};$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}\, e^{\lambda t} + \frac{qB_0}{2c}r^2\, e^{\lambda t}.$$

The final form of the Lagrangian is, then,

$$L = \frac{p_r^{\ 2}}{2m}\, e^{-\lambda t} + \frac{p_\theta^{\ 2}}{2mr^2}\, e^{-\lambda t} - \frac{q^2 B_0^{\ 2} r^2}{8mc^2}\, e^{\lambda t} - \frac{k}{2}r^2\, e^{\lambda t}. \qquad (41)$$

The Hamiltonian is

$$H_0 = \frac{p_r^{\ 2}}{2m}\, e^{-\lambda t} + \frac{1}{2mr^2}\left(p_\theta e^{\frac{-\lambda t}{2}} - \frac{qB_0 r^2}{2c} e^{\frac{\lambda t}{2}} \right)^2 + \frac{k}{2}r^2\, e^{\lambda t}; \qquad (42)$$

with

$$p_r = \left(\frac{\partial S}{\partial r}\right);$$

$$p_\theta = \left(\frac{\partial S}{\partial \theta}\right).$$

The corresponding HJE is

$$\frac{1}{2m}\, e^{-\lambda t}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial S}{\partial \theta} e^{\frac{-\lambda t}{2}} - \frac{qB_0 r^2}{2c} e^{\frac{\lambda t}{2}}\right)^2 + \frac{k}{2} r^2\, e^{\lambda t}$$

$$+\frac{\partial S}{\partial t} = 0. \qquad (43)$$

Since θ is a cyclic coordinate, the conjugate momentum must be constant: $P_\theta = \frac{\partial S}{\partial \theta} = \gamma$. To simplify, we choose $\gamma = 0.$

As a result, HJE reduces to

$$\frac{1}{2m}\, e^{-\lambda t}\left(\frac{\partial S}{\partial r}\right)^2 + \frac{q^2 B_0^2 r^2}{8mc^2} e^{\lambda t} + \frac{k}{2} r^2\, e^{\lambda t} + \frac{\partial S}{\partial t} = 0; \qquad (44)$$

or

$$\frac{1}{2m}\, e^{-\lambda t}\left(\frac{\partial S}{\partial r}\right)^2 + Cr^2\, e^{\lambda t} + \frac{\partial S}{\partial t} = 0, \qquad (45)$$

where

$$C = \left(\frac{2q^2 B_0^2 + 8mc^2 k}{16mc^2}\right).$$

Now, using a change of variables $y = re^{\frac{\lambda t}{2}}$:

$$p^2 = \left(\frac{\partial S}{\partial r}\right)^2 = \left(\frac{\partial S}{\partial y}\right)^2 e^{\lambda t}.$$

From Eq.(45), we find HJE:

$$\frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2 + Cy^2 + \frac{\partial S}{\partial t} = 0. \quad (46)$$

This equation can easily be solved to give

$$y = \sqrt{\frac{\alpha}{C}}\sin\left((\beta + t)\sqrt{\frac{2C}{m}}\right).$$

In terms of r we get,

$$r = e^{\frac{-\lambda t}{2}}\sqrt{\frac{\alpha}{C}}\sin\left((\beta + t)\sqrt{\frac{2C}{m}}\right), \quad (47)$$

4. Conclusion

In this work, dissipative systems have been investigated within the framework of the Hamilton-Jacobi method. The Hamilton-Jacobi partial differential equation for these systems has been obtained within the canonical method. The principal function S has been determined by invoking separation of variables and the chain rule, in the same manner as for regular time-independent Lagrangians. The equation of motion can then be readily obtained, thereby finding familiar results but with unfamiliar techniques.

In order to test our proposed method, we have examined three examples: the damped harmonic oscillator (together with two "variants": the RLC circuit and a viscous liquid); a system with a variable mass; and a charged particle in a magnetic field. Our formalism may shed further light on such systems as two interacting particles moving in a viscous medium, and the classical radiating electron, among others.

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