ON DOUBLE NEW INTEGRAL TRANSFORM AND DOUBLE LAPLACE TRANSFORM

Artion Kashuri, PhD Student
Department of Mathematics, Faculty of Technic Sciences
University "Ismail Qemali" of Vlora, Albania
Akli Fundo , Assoc. Prof.
Polytechnic University of Tirana, Albania
Rozana Liko, PhD Student
Department of Mathematics, Faculty of Technic Sciences
University "Ismail Qemali" of Vlora, Albania

Abstract

In this paper, a relationship between double new integral transform and double Laplace transform was establish and many other results are presented. In this paper is considered general linear telegraph equation with constant coefficients and wave equation (Hassan Eltayeb et al., 2010), (Tarig. M. Elzaki et al., 2012). The applicability of this relatively double new integral transform is demonstrated using some special functions, which arise in the solution of PDEs. First, we transform the partial differential equations to algebraic equations by using double new integral transform method and second, using the inverse double new integral transform we get the solution of PDEs.

Keywords: Double Laplace Transform, Double New Integral Transform, Single Laplace Transform, Single New Integral Transform, Inverse Double New Integral Transform

Introduction

The double new integral transform was introduced by PhD student Artion Kashuri and Associate Professor Akli Fundo to facilitate the process of solving partial differential equations in the time domain (Artion Kashuri et al., 2013), (Artion Kashuri et al., 2013). Some integral transform method such as Laplace, Sumudu transforms methods, are used to solve general nonlinear non-homogenous partial differential equation with initial conditions and use fullness of these integral transform lies in their ability to transform differential equations into algebraic equations which allows simple and systematic solution procedures. Nonlinear phenomena, that appear in many areas of scientific fields such as solid state physics, wave equation, telegraph equation, plasma physics, fluid mechanics, population models and chemical kinetics, can be modeled by nonlinear differential equations. Also the double new integral transform and some of its fundamental properties are used to solve general linear telegraph equation and wave equation with initial and boundary conditions.

The double new integral transform is defined for functions of exponential order.

In this paper is considered functions in the set F defined by:

$$F = \left\{ f(x,t) \,\middle|\, \exists M, k_1, k_2 > 0, such\ that\ |f(x,t)| \leq Me^{\frac{x+t}{k_i^2}}, if\ \ i = 1,2\ \ and\ \ (x,t) \in R_+^2 \right\} \quad (1)$$

For a given function in the set F, the constant M must be finite number, k_1, k_2 may be finite or infinite.

Definition 1.1. Let $f(x,t)$ be a function which can be expressed as a convergent infinite series and $(x,t) \in R_+^2$. Then double new integral transform is given by:

$$A(u,v) = K_2[f(x,t);(u,v)] = K_x K_t[f(x,t);(u,v)]$$

$$= \frac{1}{uv} \int_0^\infty \int_0^\infty e^{-\left(\frac{x}{u^2}+\frac{t}{v^2}\right)} f(x,t)\,dx\,dt \qquad (2)$$

The double Laplace transform of a function of two variables defined in the positive quadrant of the xt-plane is given by:

$$L_2[f(x,t);(p,q)] = \int_0^\infty \int_0^\infty e^{-(px+qt)} f(x,t)\,dx\,dt \qquad (3)$$

where p and q are the transform variables for x and t respectively, whenever the improper integral converges.

Definition 1.2. Given a function $A(u,v)$ if there is a function $f(x,t)$ that is continuous on $[0,\infty)$ and satisfies, $K_2[f(x,t);(u,v)] = A(u,v)$ then we say that $f(x,t)$ is the inverse double new integral transform of $A(u,v)$ and employ the notation:

$$f(x,t) = K_2^{-1}[A(u,v);(x,t)] \qquad (4)$$

An equivalent form of double new integral transform is:

$$A(u,v) = uv \int_0^\infty \int_0^\infty e^{-x-t} f(xu^2, tv^2) dx dt \qquad (5)$$

Definition 1.3. (Hassan Eltayeb et al., 2010)

Let $f(x,t)$ and $g(x,t)$ be continuous functions on $[0,\infty)$. Double convolution of the functions $f(x,t)$ and $g(x,t)$ exists and is defined by:

$$(f ** g)(x,t) = \int_0^t \int_0^x f(\zeta,\eta) g(x-\zeta, t-\eta) d\zeta d\eta \qquad (6)$$

Double New Integral Transform Of Some Special Functions

$(\mathbf{1})\ K_2[1;(u,v)] = uv$

$(\mathbf{2})\ K_2[(xt)^n;(u,v)] = (n!)^2 (uv)^{2n+1}$

$(\mathbf{3})\ K_2[e^{ax+bt};(u,v)] = \left(\frac{u}{1-au^2}\right)\left(\frac{v}{1-bv^2}\right)$

$(\mathbf{4})\ K_2[\sin(ax+bt);(u,v)] = uv\left[\frac{au^2+bv^2}{(1+a^2u^4)(1+b^2v^4)}\right]$

$(\mathbf{5})\ K_2[\cos(ax+bt);(u,v)] = uv\left[\frac{1-abu^2v^2}{(1+a^2u^4)(1+b^2v^4)}\right]$

$(\mathbf{6})\ K_2[(x+t)^n;(u,v)] = n! \sum_{k=0}^{n} u^{2k+1}\, v^{2(n-k)+1}$

Theorem 1.1. [Linearity of the double new integral transform**]**

Let $f(x,t)$ and $g(x,t)$ be functions whose the double new integral transform exists for
$|u| < k$ and $|v| < k$ and a, b are constants. Then,

$$K_2[af(x,t) + bg(x,t);(u,v)] = aK_2[f(x,t);(u,v)] + bK_2[g(x,t);(u,v)] \qquad (7)$$

Theorem 1.2. [Linearity of the inverse double new integral transform**]**

Assume that $K_2^{-1}[A_1(u,v);(x,t)]$, $K_2^{-1}[A_2(u,v);(x,t)]$ exists and are continuous on $[0,\infty)$ and c_1, c_2 are constants. Then,

$$K_2^{-1}[c_1A_1(u,v) + c_2A_2(u,v);(x,t)] = c_1K_2^{-1}[A_1(u,v);(x,t)] + c_2K_2^{-1}[A_2(u,v);(x,t)] \qquad (8)$$

Theorem 1.3. [Duality relation**]**
Let $f(x,t) \in F$ with double Laplace transform $L_2(s,w)$.

Then the double new integral transform $A(u,v)$ of $f(x,t)$ is given by:

$$A(u,v) = \frac{1}{uv} L_2\left[f(x,t);\left(\frac{1}{u^2},\frac{1}{v^2}\right)\right] \qquad (9)$$

Lemma 1.4. (Jean M.Tchuenche et al., 2007)

Let $f(x,t)$ and $g(x,t)$ be two real-valued functions whose the double new integral transform $A(u,v)$ and $G(u,v)$ exists for $|u| < k$ and $|v| < k$ and a, b are positive constants. Then,

$$(1)\ K_2[f(ax)g(bt);(u,v)] = \left(\frac{1}{\sqrt{a}}A(\sqrt{a}u)\right)\left(\frac{1}{\sqrt{b}}G(\sqrt{b}v)\right)$$

$$(2)\ K_2[f(ax,bt);(u,v)] = \frac{1}{\sqrt{ab}}A[\sqrt{a}u,\sqrt{b}v]$$

Theorem 1.5. [Fundamental properties of the double new integral transform of partial derivatives**]**

Let $A(u,v)$ be the double new integral transform of $f(x,t)$. Then,

$$(1)\ K_2\left[\frac{\partial f(x,t)}{\partial x};(u,v)\right] = \frac{A(u,v)}{u^2} - \frac{A(0,v)}{u}$$

$$(2)\ K_2\left[\frac{\partial^2 f(x,t)}{\partial x^2};(u,v)\right] = \frac{A(u,v)}{u^4} - \frac{A(0,v)}{u^3} - \frac{1}{u}\frac{\partial A(0,v)}{\partial x}$$

$$(3)\ K_2\left[\frac{\partial^n f(x,t)}{\partial x^n};(u,v)\right] = \frac{A(u,v)}{u^{2n}} - \sum_{k=0}^{n-1}\frac{1}{u^{2(n-k)-1}}\frac{\partial^k A(0,v)}{\partial x^k}$$

$$(4)\ K_2\left[\frac{\partial f(x,t)}{\partial t};(u,v)\right] = \frac{A(u,v)}{v^2} - \frac{A(u,0)}{v}$$

$$(5)\ K_2\left[\frac{\partial^2 f(x,t)}{\partial t^2};(u,v)\right] = \frac{A(u,v)}{v^4} - \frac{A(u,0)}{v^3} - \frac{1}{v}\frac{\partial A(u,0)}{\partial t}$$

$$(6)\ K_2\left[\frac{\partial^n f(x,t)}{\partial t^n};(u,v)\right] = \frac{A(u,v)}{v^{2n}} - \sum_{k=0}^{n-1}\frac{1}{v^{2(n-k)-1}}\frac{\partial^k A(u,0)}{\partial t^k}$$

$$(7)\ K_2\left[\frac{\partial^2 f(x,t)}{\partial x\partial t};(u,v)\right] = \frac{1}{u^2v^2}A(u,v) - \frac{1}{u^2v}A(u,0) - \frac{1}{v^2u}A(0,v) + \frac{1}{uv}f(0,0)$$

Theorem 1.6. [Translation theorem**]**

Let $f(x,t) \in F$ with the double new integral transform $A(u,v)$. Then,

$$K_2[e^{ax+bt}f(x,t);(u,v)] = \left(\frac{1}{\sqrt{1-au^2}}\right)\left(\frac{1}{\sqrt{1-bv^2}}\right)A\left[\frac{u}{\sqrt{1-au^2}},\frac{v}{\sqrt{1-bv^2}}\right] \quad (10)$$

Theorem 1.7. [Double convolution theorem**]**

Let $f(x,t)$ and $g(x,t)$ be defined in F having double new integral transforms $A(u,v)$ and $G(u,v)$. Then the double new integral transform of the double convolution of $f(x,t)$ and $g(x,t)$ is given by:

$$K_2[(f ** g)(x,t);(u,v)] = (uv)A(u,v)G(u,v) \quad (11)$$

Theorem 1.8. [Initial and Final value theorem**]**

Let $f(x,t) \in F$ and suppose that either $\lim_{(x,t)\to(0,0)} \sqrt{xt}\, f(x,t)$ or $\lim_{(x,t)\to(\infty,\infty)} \sqrt{xt}\, f(x,t)$ exists. Then,

$$(\boldsymbol{a}) \lim_{(u,v)\to(0,0)} A(u,v) = \pi\left[\lim_{(x,t)\to(0,0)} \sqrt{xt}\, f(x,t)\right]$$

$$(\boldsymbol{b}) \lim_{(u,v)\to(\infty,\infty)} A(u,v) = \pi\left[\lim_{(x,t)\to(\infty,\infty)} \sqrt{xt}\, f(x,t)\right]$$

Theorem 1.9. Let $A'(u,v)$ denote a double new integral transform of the definite double integral of $f(x,t)$.

$$h(x,t) = \int_0^t \int_0^x f(\zeta,\eta)\, d\zeta\, d\eta$$

Then,

$$A'(u,v) = K_2[h(x,t);(u,v)] = (uv)^2 A(u,v) \quad (12)$$

Applications

As stated in the introduction of this paper, the double new integral transform can be used as an effective tool for solving partial differential equation. The following two examples illustrate the use of the double new integral transform. It is well known that in order to obtain the solution of partial differential equations by integral transform methods we need the following two steps:

- Firstly, we transform the partial differential equations to algebraic equations by using double new integral transform method.

- Secondly, on using inverse double new integral transform we get the solution of PDEs.

Consider the general linear telegraph equation (Tarig. M. Elzaki et al., 2012) in the form:

$U_{tt} + aU_t + bU = c^2 U_{xx}$, where a, b, c are constants (1)

with initial conditions:

$U(x,0) = f_1(x), \quad U_t(x,0) = g_1(x)$ (2)

and boundary conditions:

$U(0,t) = f_2(t), \quad U_x(0,t) = g_2(t)$ (3)

Solution:

$$K_2[U_{tt}(x,t);(u,v)] = \frac{A(u,v)}{v^4} - \frac{A(u,0)}{v^3} - \frac{1}{v}\frac{\partial A(u,0)}{\partial t}$$

$$K_2[U_t(x,t);(u,v)] = \frac{A(u,v)}{v^2} - \frac{A(u,0)}{v}; \quad K_2[U(x,t);(u,v)] = A(u,v)$$

$$K_2[U_{xx}(x,t);(u,v)] = \frac{A(u,v)}{u^4} - \frac{A(0,v)}{u^3} - \frac{1}{u}\frac{\partial A(0,v)}{\partial x}$$

$$A(u,0) = F_1(u) = \frac{1}{u}\int_0^\infty e^{-\frac{x}{u^2}} U(x,0)dx = \frac{1}{u}\int_0^\infty e^{-\frac{x}{u^2}} f_1(x)dx \qquad (4)$$

$$A(0,v) = F_2(v) = \frac{1}{v}\int_0^\infty e^{-\frac{t}{v^2}} U(0,t)dt = \frac{1}{v}\int_0^\infty e^{-\frac{t}{v^2}} f_2(t)dt \qquad (5)$$

$$\frac{\partial A(u,0)}{\partial t} = G_1(u) = \frac{1}{u}\int_0^\infty e^{-\frac{x}{u^2}} \frac{\partial U(x,0)}{\partial t} dx = \frac{1}{u}\int_0^\infty e^{-\frac{x}{u^2}} g_1(x)dx \qquad (6)$$

$$\frac{\partial A(0,v)}{\partial x} = G_2(v) = \frac{1}{v}\int_0^\infty e^{-\frac{t}{v^2}} \frac{\partial U(0,t)}{\partial x} dt = \frac{1}{v}\int_0^\infty e^{-\frac{t}{v^2}} g_2(t)dt \qquad (7)$$

Take the double new integral transform of Eq. (1) and single new integral transform of conditions, then we have:

$$\left(\frac{A(u,v)}{v^4} - \frac{A(u,0)}{v^3} - \frac{1}{v}\frac{\partial A(u,0)}{\partial t}\right) + a\left(\frac{A(u,v)}{v^2} - \frac{A(u,0)}{v}\right) + bA(u,v)$$
$$= c^2\left(\frac{A(u,v)}{u^4} - \frac{A(0,v)}{u^3} - \frac{1}{u}\frac{\partial A(0,v)}{\partial x}\right)$$

Substituting initial and boundary conditions $(2) - (3)$ in equation above we have:

$$\frac{A(u,v)}{v^4} - \frac{1}{v^3}F_1(u) - \frac{1}{v}G_1(u) + \frac{a}{v^2}A(u,v) - \frac{a}{v}F_1(u) + bA(u,v)$$

$$-\frac{c^2}{u^4}A(u,v)+\frac{c^2}{u^3}F_2(v)+\frac{c^2}{u}G_2(v)=0$$

After some simple algebraic operations we get:

$$A(u,v)=\frac{F_1(u)(vu^3+au^3v^3)-c^2v^4F_2(v)-c^2u^2v^4G_2(v)+u^3v^3G_1(u)}{u^3+au^3v^2+bu^3v^4-c^2u^3}$$
$$=N(u,v) \qquad (8)$$

Take invers double new integral transform to obtain the solution of general linear telegraph Eq.(1) in the form:

$$U(x,t)=K_2^{-1}[A(u,v);(x,t)]=K_2^{-1}[N(u,v);(x,t)]=D(x,t) \qquad (9)$$

Assumed that the double new integral transform exists.

Example 1.1. Consider the linear telegraph equation in the form:

$$U_{xx}=U_{tt}+U_t+U \qquad (10)$$

with initial conditions:

$$U(x,0)=f_1(x)=e^x, \quad U_t(x,0)=g_1(x)=-e^x \qquad (11)$$

and boundary conditions:

$$U(0,t)=f_2(t)=e^{-t}, \quad U_x(0,t)=g_2(t)=e^{-t} \qquad (12)$$

Solution:

Take the double new integral transform of Eq. (10) and single new integral transform of conditions, then we have:

$$\frac{A(u,v)}{u^4}-\frac{A(0,v)}{u^3}-\frac{1}{u}\frac{\partial A(0,v)}{\partial x}$$
$$=\frac{A(u,v)}{v^4}-\frac{A(u,0)}{v^3}-\frac{1}{v}\frac{\partial A(u,0)}{\partial t}+\frac{A(u,v)}{v^2}-\frac{A(u,0)}{v}+A(u,v)$$

$$A(u,0)=\frac{u}{1-u^2};\quad A(0,v)=\frac{v}{1+v^2};\quad \frac{\partial A(u,0)}{\partial t}=-\frac{u}{1-u^2};\quad \frac{\partial A(0,v)}{\partial x}=\frac{v}{1+v^2}$$

Substituting initial and boundary conditions $(11)-(12)$ in equation above we have:

$$\frac{A(u,v)}{u^4}-\frac{1}{u^3}\left(\frac{v}{1+v^2}\right)-\frac{1}{u}\left(\frac{v}{1+v^2}\right)$$
$$=\frac{1}{v^4}A(u,v)-\frac{1}{v^3}\left(\frac{u}{1-u^2}\right)+\frac{1}{v}\left(\frac{u}{1-u^2}\right)+\frac{1}{v^2}A(u,v)-\frac{1}{v}\left(\frac{u}{1-u^2}\right)$$
$$+A(u,v)$$

After some simple algebraic operations we get:

$$A(u,v)=\frac{uv}{(1-u^2)(1+v^2)} \qquad (13)$$

Take invers double new integral transform to obtain the solution of linear telegraph Eq. (10) in the form:

$$U(x,t) = K_2^{-1}[A(u,v);(x,t)] = e^{x-t} \quad (14)$$

Example 1.2. Consider wave equation in the form:

$$U_{tt} - U_{xx} = 3(e^{x+2t} - e^{2x+t}) \qquad (x,t) \in R_+^2 \quad (15)$$

with initial conditions:

$$U(x,0) = f_1(x) = e^{2x} + e^x, \quad U_t(x,0) = g_1(x) = e^{2x} + 2e^x \quad (16)$$

and boundary conditions:

$$U(0,t) = f_2(t) = e^t + e^{2t}, \quad U_x(0,t) = g_2(t) = 2e^t + e^{2t} \quad (17)$$

Solution

Take the double new integral transform of Eq. (15) and single new integral transform of conditions, then we have:

$$\left(\frac{A(u,v)}{v^4} - \frac{A(u,0)}{v^3} - \frac{1}{v}\frac{\partial A(u,0)}{\partial t}\right) - \left(\frac{A(u,v)}{u^4} - \frac{A(0,v)}{u^3} - \frac{1}{u}\frac{\partial A(0,v)}{\partial x}\right) = 3\left(\frac{uv}{(1-u^2)(1-2v^2)} - \frac{uv}{(1-2u^2)(1-v^2)}\right) \quad (18)$$

$$A(u,0) = \frac{u}{1-2u^2} + \frac{u}{1-u^2}; \quad A(0,v) = \frac{v}{1-v^2} + \frac{v}{1-2v^2}$$

$$\frac{\partial A(u,0)}{\partial t} = \frac{u}{1-2u^2} + \frac{2u}{1-u^2}; \quad \frac{\partial A(0,v)}{\partial x} = \frac{2v}{1-v^2} + \frac{v}{1-2v^2}$$

Substituting initial and boundary conditions (16)– (17) in Eq. (18) and after some simple algebraic operations we have:

$$A(u,v) = \frac{uv}{(1-2u^2)(1-v^2)} + \frac{uv}{(1-u^2)(1-2v^2)} \quad (19)$$

Take invers double new integral transform to obtain the solution of wave Eq. (15) in the form:

$$U(x,t) = K_2^{-1}[A(u,v);(x,t)] = e^{2x+t} + e^{x+2t} \quad (20)$$

Conclusion

In this paper, we studied the double new integral transform and apply them to solve general linear telegraph equation and wave equation. The applicability of this relatively double new integral transform is demonstrated using some special functions. Double new integral transform method is strong method to solve such PDEs. It may be concluded that double new integral transform is very powerful and efficient in finding the analytical solution for a wide class of initial boundary value problems. The connection of the double new integral transform with double Laplace transform goes much deeper and we can find other relations of the double new integral transform by this connection.

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