# CANONICAL QUANTIZATION OF DISSIPATIVE SYSTEMS

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**Abstract**

The canonical method is invoked to quantize dissipative systems using the WKB approximation. The wave function is constructed such that its phase factor is simply Hamilton's principal function. The energy eigenvalue is found to be in exact agreement with the classical case. To demonstrate our approach, the three examples considered in our previous work (*ESJ* 9(30), 70-81, 2013) are quantized in detail: the damped harmonic oscillator, a system with a variable mass, and a charged particle in a magnetic field.

**Keywords:** Hamilton-Jacobi Equation, dissipative systems, WKB approximation

## 1. Introduction

Many advanced methods of classical mechanics deal with only conservative systems, although all natural processes in the physical world are nonconservative. Whether treated classically or quantum-mechanically, and whether viewed macroscopically or microscopically, the physical world manifests different kinds of dissipation and irreversibility. Mostly ignored in analytic techniques, dissipation appears in friction, Brownian motion, inelastic scattering, electric resistance, and many other processes in nature.

Several attempts have been made to incorporate nonconservative forces into Lagrangian and Hamiltonian formulations; but those attempts could not give a completely consistent physical framework for these forces. The Rayleigh dissipation function, invoked when the frictional force is proportional to the velocity (Goldstein,1980), was the first to describe frictional forces in the Lagrangian formulation. However, in that case, another scalar function was needed, in addition to the Lagrangian, to specify the equations of motion. At the same time, this function did not appear in the Hamiltonian. Accordingly, the whole process was of no use when it was attempted to quantize nonconservative systems.

The most substantive work in this context was that of (Riewe, 1996,1997), who used fractional derivatives to study nonconservative systems and was able to generalize the Lagrangian and other classical functions to take into account nonconservative effects.

As a sequel to Riewe`s work, (Rabei,2004) used Laplace transforms of fractional integrals and fractional derivatives to develop a general formula for the potential of an arbitrary force, conservative or nonconservative. This led directly to the consideration of dissipative effects in Lagrangian and Hamiltonian formulations.

Most recently, dissipative systems were investigated using the Hamilton-Jacobi equation (HJE) (Jarab'ah,2013). This equation was solved using the separation-of-variables technique. The corresponding principal function was found. The equation of motion could then be derived from this function, which represented the energy of the system, in terms of the generalized coordinates and momenta. This, in turn, could constitute a basis for the so-called canonical quantization using the WKB approximation, thereby obtaining the corresponding Hamiltonian and Schrödinger's equation (Das,2005).

The purpose of the present work is indeed to quantize dissipative systems using the WKB approximation. The paper is organized as follows. In Section 2, our Hamilton-Jacobi method for dissipative systems is reviewed briefly. In Section 3, the quantization of such systems using the WKB approximation is outlined. In Section 4, the three dissipative systems examined in our previous work (Jaraba'ah,2005) -- namely, the damped harmonic oscillator (together with the RLC circuit and a viscous liquid); a system with a variable mass; and a charged particle in a magnetic field – are quantized within this approximation. Finally, in Section 5, the work closes with some concluding remarks.

## 2. Brief Review of the Hamilton-Jacobi Formalism

We start with the Lagrangian

$$L_0 = L(q, \dot{q})\, e^{\lambda t}, \tag{1}$$

$\lambda$ : being some constant.

As usual, the generalized momentum, defined by [7]

$$p_i = \frac{\partial L}{\partial \dot{q}_i},$$

gives the corresponding Hamiltonian $H_o$ in terms of the generalized coordinates q and generalized momenta p as

$$H_0 = p_i \dot{q}_i - L_0 \equiv H_0(q_i, p_i).$$

(2)

Therefore, the corresponding HJE of Eq. (2) will be of the form

$$H_0(q_1, q_2, \ldots, q_N; \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \ldots, \frac{\partial S}{\partial q_N}; t) + \frac{\partial S}{\partial t} = 0,$$

(3)

where

$$p_i = \frac{\partial S}{\partial q_i}.$$

Here the generalized momenta do not appear in Eq. (3), except as derivatives of Hamilton's principal function S, which is a function of the N generalized coordinates $q_1, q_2, \ldots, q_N$ and the time t.

Since $L \equiv T - V$ is the physical Lagrangian of the system, T being the kinetic energy and V the potential energy, it follows that $H_0$ is the physical Hamiltonian representing the system's total energy: T+V (Goldstein,1980).

The resulting action S is

$$S = \int e^{\lambda t} L dt = \int (p\dot{q} - H_0) dt.$$

(4)

Now, if $S(q_1, q_2, ..., q_N; \alpha_1, \alpha_2, ..., \alpha_N)$ is a complete integral of HJE, the integrals of Hamilton's equations of motion will be given by (Goldstein,1980)

$$\frac{\partial S}{\partial \alpha_j} = \beta_j;$$

in addition to

$$p_j = \frac{\partial S}{\partial q_j},$$

$\beta_j$ being some constants.

To construct HJE, we may write S in a separable form as(Goldstein,1980)

$$S(q, \alpha, t) = W(q, \alpha) + f(t), \tag{5}$$

Where the time-independent function W(q,α) is the so-called Hamilton's characteristic function.

Differentiating Eq. (5) with respect to t, we find that

$$\frac{\partial S}{\partial t} = \frac{\partial f}{\partial t}. \tag{6}$$

From Eq. (3), it follows that

$$\frac{\partial f}{\partial t} = -H_0. \tag{7}$$

The left-hand side of this equation depends on t alone; whereas the right-hand side depends on q alone. Each side must then be equal to a constant independent of both q and t. Therefore, the time derivative $\partial S/\partial t$ in HJE must be a constant, usually denoted by (-α).

Thus,

$$S(q,\alpha,t) = W(q,\alpha) - \alpha t. \tag{8}$$

It follows that

$$\mathrm{H}_0\left(\mathrm{q}, \frac{\partial \mathrm{W(q)}}{\partial \mathrm{q}}\right) = \alpha.$$

**3. Quantization Using the WKB Approximation**

It is well known that HJE for dissipative systems leads naturally to the semiclassical approximation, namely, WKB (Rabei,2002). This is a basic technique for obtaining an approximate solution to Schrödinger's equation. It has been used since the early days of quantum mechanics for determining the approximate spectra of bound-state problems for certain potentials (Landau,1958,Alonso,1973, Griffth,1995). The quantization of classical systems can be achieved by the canonical method. Starting with the Hamiltonian, one raises the coordinates and momenta to the status of operators and carries out the quantization (Hasse,1975,Razavy,1977).

For dissipative systems, the Hamiltonian operator $\hat{H}_0$, corresponding to the classical function $H_0$, is found by using the conventional quantization rule and replacing the canonical momentum with $p = \frac{\hbar}{i}\frac{\partial}{\partial q}$. The Schrödinger equation will then be

$$i\hbar \frac{\partial \psi(q,t)}{\partial t} = \left[ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q) \right]\psi(q,t); \tag{9}$$

or, more explicitly,

$$i\hbar\frac{\partial}{\partial t}\psi = \hat{H}_0\psi;$$

Where

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q).$$

The quantization procedure is realized as follows:

Using the familiar complex form of the wave function (Merzbacher,1961)

$$\psi(q,t) = \exp\left(\frac{iS(q,t)}{\hbar}\right),$$

the amplitude being set to unity for convenience, we have

$$-\frac{\partial S}{\partial t}\psi = \left[\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 - \frac{i\hbar}{2m}\frac{\partial^2 S}{\partial q^2} + V\right]\psi.$$

Since $\psi \neq 0$, this leads to

$$-\frac{\partial S}{\partial t} = \left[\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 - \frac{i\hbar}{2m}\frac{\partial^2 S}{\partial q^2} + V\right].$$

(10) In the limit $\hbar \to 0$:

$$-\frac{\partial S}{\partial t} = \left(\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V\right).$$

(11)

The function $S$ is written as $S(q,\alpha,t) = S(q,\alpha) - \alpha t.$

Differentiating this equation with respect to t and q, and inserting the result in Eq. (11), we end up with

$$\alpha = \left( \frac{1}{2m}\left( \frac{\partial S}{\partial q} \right)^2 + V \right).$$

(12)

Rearranging and integrating Eq. (12), we finally obtain

$$S(q) = \int \sqrt{2m\left(\alpha - V(q)\right)}dq.$$

This satisfies the canonical relation $(\hat{H}_0 + \hat{p}_0)\psi = 0$,

where $\hat{p}_0 = \frac{\hbar}{i}\frac{\partial}{\partial t}$; and our quantization is complete.

## 4. Examples

### *4.1 Damped Harmonic Oscillator*

The following Lagrangian is suitable for this system in one dimension (Bateman,1931):

$$L_0(q, \dot{q}, t) = \left( \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega^2 q^2 \right) e^{\lambda t},$$

(13)

Where m is the mass, and $\omega$ the frequency.

The linear momentum is given by

$$p = \frac{\partial L_0}{\partial \dot{q}} = m\dot{q}\ e^{\lambda t}.$$

Using the standard form of the Hamiltonian:

$$H_0 = p\dot{q} - L_0,$$

we find

$$H_0 = \frac{p^2}{2m} e^{-\lambda t} + \frac{1}{2} m\omega^2 q^2 e^{\lambda t}.$$

If we make the substitution

$$y = q \, e^{\frac{\lambda t}{2}},$$

then $H_0$ can be obtained as

$$H_0 = \frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2 + \frac{1}{2} m\omega^2 y^2.$$

(14)

The corresponding HJE takes the form

$$\frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2 + \frac{1}{2} m\omega^2 y^2 + \frac{\partial S}{\partial t} = 0.$$

(15)

It is possible to propose that [1]

$$S(y,\alpha,t) = W(y,\alpha) - \alpha t.$$

(16)

Differentiating Eq. (16), first with respect to time and then with respect to the coordinate y, and substituting the results into Eq. (15), we get

$$\frac{1}{2m}\left(\frac{\partial W}{\partial y}\right)^2 + \frac{1}{2} m\omega^2 y^2 - \alpha = 0;$$

(17)

so that

$$W = \int \sqrt{2m\alpha - m^2\omega^2 y^2}\, dy.$$

We finally obtain for the function S:

$$S = \int \sqrt{2m\alpha - m^2\omega^2 y^2}\, dy - \alpha t.$$

(18)

We are now ready to obtain the equations of motion. Making use of the canonical transformation, we find

$$\beta \equiv \frac{\partial S}{\partial \alpha} = -t + \int \frac{m}{\sqrt{2m\alpha - m^2\omega^2 y^2}} dy = -t + \frac{1}{\omega}\int \frac{dy}{\sqrt{\frac{2\alpha}{m\omega^2} - y^2}}..$$

Then

$$y = \sqrt{\frac{2\alpha}{m\omega^2}} \sin((\beta + t)\omega).$$

Finally,

$$q = \sqrt{\frac{2\alpha}{m\omega^2}} \sin((\beta + t)\omega)\, e^{\frac{-\lambda t}{2}},$$

(19)

and

$$p = \sqrt{2m\alpha - m^2\omega^2 y^2}\; e^{\frac{\lambda t}{2}}.$$

(20)

We are now ready to quantize our example. Treating $H_0$ as an operator, we have

$$\hat{H}_0 = \frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega^2 y^2.$$

The Schrödinger equation for a damped harmonic oscillator reads

$$(\hat{H}_0 + \hat{p}_0)\psi = 0;$$

(21) or

$$\left(\hat{p}_0 + \frac{\hat{p}_y^2}{2m} + \frac{1}{2}m\omega^2 y^2\right)\psi = 0,$$

(22)

with

$$\hat{p}_0 \equiv \frac{\hbar}{i}\frac{\partial}{\partial t};$$

$$\hat{p}_y \equiv \frac{\hbar}{i}\frac{\partial}{\partial y};$$

$$\psi \equiv e^{\frac{iS}{\hbar}}.$$

Thus, the Hamiltonian becomes

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial t} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2}m\omega^2 y^2\right) e^{\frac{iS}{\hbar}} = 0.$$

(23) In fact,

$$\frac{\partial}{\partial t} e^{\frac{iS(y,t)}{\hbar}} = \frac{i}{\hbar} e^{\frac{iS(y,t)}{\hbar}} \frac{\partial S}{\partial t} = \frac{i}{\hbar}(-\alpha)\psi;$$

(24)

and the partial derivative with respect to y is

$$\frac{\partial}{\partial y} e^{\frac{iS(y,t)}{\hbar}} = \frac{i}{\hbar} e^{\frac{iS(y,t)}{\hbar}} \frac{\partial S}{\partial y} = \frac{i}{\hbar}\left(\frac{\partial S}{\partial y}\right)\psi;$$

(25)

so that the second derivative is

$$\frac{\partial^2}{\partial y^2} e^{\frac{iS(y,t)}{\hbar}} = \frac{\partial}{\partial y}\left(\frac{i}{\hbar} e^{\frac{iS(y,t)}{\hbar}} \frac{\partial S}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{i}{\hbar}\left(\frac{\partial S}{\partial y}\right)\psi\right);$$

or

$$\frac{\partial^2}{\partial y^2} e^{\frac{iS(y,t)}{\hbar}} = \frac{i}{\hbar}\frac{\partial S}{\partial y}\frac{\partial \psi}{\partial y} + \frac{i}{\hbar}\psi\frac{\partial^2 S}{\partial y^2}.$$

(26)

With Eq. (25), this becomes

$$\frac{\partial^2}{\partial y^2} e^{\frac{iS(y,t)}{\hbar}}$$

$$= \frac{i}{\hbar}\frac{\partial S}{\partial y}\frac{i}{\hbar}\frac{\partial S}{\partial y}\psi + \frac{i}{\hbar}\psi\frac{\partial^2 S}{\partial y^2} = \frac{-1}{\hbar^2}\psi\left(\frac{\partial S}{\partial y}\right)^2 + \frac{i}{\hbar}\psi\frac{\partial^2 S}{\partial y^2}. \quad (27)$$

Putting Eqs. (24) and (27) into (23), we get

$$\left(-\alpha+\frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2-\frac{i\hbar}{2m}\frac{\partial^2 S}{\partial y^2}+\frac{1}{2}m\omega^2 y^2\right)\psi=0.$$

(28) Since $\psi \neq 0$, this leads to

$$-\alpha+\frac{1}{2m}\left(\frac{\partial S}{\partial y}\right)^2-\frac{i\hbar}{2m}\frac{\partial^2 S}{\partial y^2}+\frac{1}{2}m\omega^2 y^2=0.$$

(29)

Taking the formal limit ħ→0, and recalling that $S=\int\sqrt{2m\alpha-m^2\omega^2 y^2}\,dy-\alpha t$, we obtain the classical HJE:

$$-\alpha+\frac{1}{2m}(2m\alpha-m^2\omega^2 y^2)+\frac{1}{2}m\omega^2 y^2=0.$$

This satisfies Eq. (21). The quantization of the damped harmonic oscillator is now complete.

One can follow the same steps outlined in this example to study other dissipative systems, such as the RLC circuit and a viscous liquid, as follows:

For the RLC circuit, an appropriate Lagrangian is (Pain,2005)

$$L_0(Q,\dot{Q},t)=\left(\frac{1}{2}L\dot{Q}^2-\frac{Q^2}{2C}\right)e^{\lambda t}.$$

It follows that

$$H_0 = \frac{1}{2L}\left(\frac{\partial S}{\partial y}\right)^2 + \frac{y^2}{2C}.$$

The HJ function can be obtained as

$$S = \int \sqrt{2L\alpha - \frac{y^2 L}{C}}dy - \alpha t.$$

The equations of motion are

$$Q = A\sin\left((\beta + t)\sqrt{\frac{1}{CL}}\right) e^{\frac{-\lambda t}{2}};$$

$$p = \sqrt{2L\alpha - \frac{y^2 L}{C}}\; e^{\frac{\lambda t}{2}}.$$

Using the same steps for quantization, we have the following result:

$$-\alpha + \frac{1}{2L}\left(2L\alpha - \frac{y^2 L}{C}\right) + \frac{y^2}{2C} = 0,$$

which satisfies the quantization condition.

For a viscous liquid in a tube, we have the following Lagrangian(Pain,2005):

$$L_0(q,\dot{q},t)=\left(\frac{1}{2}l\dot{q}^2-gq^2\right)e^{\lambda t},$$

where $l$ is the length of the liquid column, g is the gravitational acceleration taken here as constant, and q represents the variations in the liquid height.

Its Hamiltonian is given by

$$H_0=\frac{1}{2l}\left(\frac{\partial S}{\partial y}\right)^2+gy^2.$$

The HJ function can be obtained as

$$S=\int\sqrt{2l\alpha-2gly^2}\,dy-\alpha t.$$

Finally, the equations of motion are

$$q=\sqrt{\frac{\alpha}{g}}\sin\left((\beta+t)\sqrt{\frac{2g}{l}}\right)e^{\frac{-\lambda t}{2}};$$

$$p=\sqrt{2l\alpha-2gly^2}\,e^{\frac{\lambda t}{2}}.$$

The quantization result is

$$-\alpha+\frac{1}{2l}(2l\alpha-2gly^2)+gy^2=0.$$

*4.2 System with a Variable Mass*

A suitable Lagrangian for this system is(Razavy,2005):

$$L(q,\dot{q},t) = \left(\frac{1}{2}m\dot{q}^2 - mgq\right). \tag{30}$$

Suppose that the mass changes with time according to

$$m = m_0\, e^{\lambda t}.$$

Then

$$L_0(q,\dot{q},t) = \left(\frac{1}{2}m_0\dot{q}^2 - m_0 gq\right) e^{\lambda t}. \tag{31}$$

Clearly, the damping factor here arises from the variation of the mass with time.

The linear momentum is given by

$$p = m_0\dot{q}\, e^{\lambda t}.$$

The usual treatment gives

$$H_0 = \frac{p^2}{2m_0}\, e^{-\lambda t} + m_0 gq\, e^{\lambda t}; \tag{32}$$

and HJE is

$$\frac{p^2}{2m_0}\, e^{-\lambda t} + m_0 gq\, e^{\lambda t} + \frac{\partial S}{\partial t} = 0.$$

Further, the principal function takes the form

$$S(q,t) = qN(t) + D(t).$$

(33)

So one gets

$$\frac{\partial S}{\partial t} = qN'(t) + D'(t).$$

(34)

With $p = \dfrac{\partial S}{\partial q}$, we have

$$p^2 = \left(\frac{\partial S}{\partial q}\right)^2 = \left(N(t)\right)^2.$$

(35)

The corresponding HJE takes the form

$$\frac{1}{2m_0}\left(N(t)\right)^2 e^{-\lambda t} + m_0 gq\, e^{\lambda t} + qN'(t) + D'(t) = 0.$$

(36)

Matching powers of q then integrating, we get

$$N(t) = -m_0 g\frac{e^{\lambda t}}{\lambda} + N_0;$$

$$D(t) = -m_0 g^2 \frac{e^{\lambda t}}{2\lambda^3} + N_0^2 \frac{e^{-\lambda t}}{2m_0\lambda} + \frac{gN_0 t}{\lambda} + D_0.$$

From Eq. (33), it follows that

$$S = -m_0 g q \frac{e^{\lambda t}}{\lambda} + N_0 q - m_0 g^2 \frac{e^{\lambda t}}{2\lambda^3} + N_0^2 \frac{e^{-\lambda t}}{2m_0\lambda} + \frac{gN_0 t}{\lambda} + D_0 .$$

(37)

Then

$$q = \beta - N_0 \frac{e^{-\lambda t}}{m_0\lambda} - \frac{gt}{\lambda},$$

(38)

and

$$p = \frac{\partial S}{\partial q} = -m_0 g \frac{e^{\lambda t}}{\lambda} + N_0 .$$

(39)

Now, using $H_0$ and p as operators, we have the Schrödinger equation

$$\left( \frac{\hbar}{i}\frac{\partial}{\partial t} - \frac{\hbar^2}{2m_0} e^{-\lambda t} \frac{\partial^2}{\partial q^2} + m_0 g q e^{\lambda t} \right) e^{\frac{iS}{\hbar}} = 0 .$$

(40)

But

$$\frac{\partial}{\partial t} e^{\frac{iS(q,t)}{\hbar}} = \frac{i}{\hbar}\begin{pmatrix} -m_0 gq e^{\lambda t} - m_0 g^2 \frac{e^{\lambda t}}{2\lambda^2} - \\ N_0^2 \frac{e^{-\lambda t}}{2m_0} + \frac{gN_0}{\lambda} \end{pmatrix}\psi,$$

and

$$\frac{\partial^2}{\partial q^2} e^{\frac{iS(q,t)}{\hbar}} = \frac{-1}{\hbar^2}\psi\left(\frac{\partial S}{\partial q}\right)^2 + \frac{i}{\hbar}\psi\frac{\partial^2 S}{\partial q^2}.$$

Putting the above equations into Eq. (40), it is easy to show that in the limit $\hbar\to 0$, we will satisfy the canonical relation; and our quantization is complete.

### *4.3 A Charged Particle in a Magnetic Field*

As a final example, consider the motion in two dimensions of a charged particle under the influence of a central force potential, $V=kr^2/2$, as well as an external constant magnetic field perpendicular to the plane of motion: $\vec{B} = B_0\hat{k}$.

The vector potential is

$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{r} = \frac{1}{2}B_0(-y\hat{i} + x\hat{j}).$$

The Lagrangian is (Goldstein,1980)

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{q}{c}(\vec{v}.\vec{A}) - \frac{k}{2}(x^2 + y^2).$$

In the presence of damping effects, the Lagrangian becomes

$$L_0 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\ e^{\lambda t} + \frac{q}{c}(\vec{v}.\vec{A})\ e^{\lambda t} - \frac{k}{2}(x^2 + y^2)\ e^{\lambda t}.$$

(41)

With $\vec{v} = \dot{x}\hat{i} + \dot{y}\hat{j}$:

$$L_0 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)\ e^{\lambda t} + \frac{qB_0}{2c}(x\dot{y} - y\dot{x})\ e^{\lambda t} - \frac{k}{2}(x^2 + y^2)\ e^{\lambda t}.$$

(42)

To simplify, polar coordinates are used:

$$x = r\cos\theta;$$
$$y = r\sin\theta.$$

Then Eq. (42) becomes

$$L_0 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)\ e^{\lambda t} + \frac{qB_0}{2c}r^2\dot{\theta}\ e^{\lambda t} - \frac{k}{2}r^2\ e^{\lambda t}.$$

The conjugate momenta are

$$p_r = \frac{\partial L_0}{\partial \dot{r}} = m\dot{r}\ e^{\lambda t};$$

$$p_\theta = \frac{\partial L_0}{\partial \dot{\theta}} = mr^2\dot{\theta}\ e^{\lambda t} + \frac{qB_0}{2c}r^2\ e^{\lambda t}.$$

The final form of the Lagrangian is

$$L_0 = \frac{{p_r}^2}{2m}\ e^{-\lambda t} + \frac{{p_\theta}^2}{2mr^2}\ e^{-\lambda t} - \frac{q^2{B_0}^2r^2}{8mc^2}\ e^{\lambda t} - \frac{k}{2}r^2\ e^{\lambda t}.$$

The Hamiltonian is

$$H_0 = \frac{p_r^{\ 2}}{2m} e^{-\lambda t} + \frac{1}{2mr^2}\left( p_\theta e^{\frac{-\lambda t}{2}} - \frac{qB_0 r^2}{2c} e^{\frac{\lambda t}{2}} \right)^2 + \frac{k}{2} r^2 \ e^{\lambda t};$$

(43)

or, with

$$p_r = \left(\frac{\partial S}{\partial r}\right); \qquad p_\theta = \left(\frac{\partial S}{\partial \theta}\right).$$

the corresponding HJE is

$$\frac{1}{2m} e^{-\lambda t} \left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{2mr^2}\left( \frac{\partial S}{\partial \theta} e^{\frac{-\lambda t}{2}} - \frac{qB_0 r^2}{2c} e^{\frac{\lambda t}{2}} \right)^2 + \frac{k}{2} r^2 \ e^{\lambda t}$$

$$+ \frac{\partial S}{\partial t} = 0.$$

Since $\theta$ is a cyclic coordinate, the conjugate momentum must be constant: $P_\theta = \frac{\partial S}{\partial \theta} = \gamma$. To simplify, we choose $\gamma = 0.$

The corresponding HJE then reduces to

$$\frac{1}{2m} e^{-\lambda t} \left(\frac{\partial S}{\partial r}\right)^2 + \frac{q^2 B_0^2 r^2}{8mc^2} e^{\lambda t} + \frac{k}{2} r^2 \ e^{\lambda t} + \frac{\partial S}{\partial t} = 0;$$

(44)

or

$$\frac{1}{2m}\ e^{-\lambda t}\left(\frac{\partial S}{\partial r}\right)^2 + Cr^2\ e^{\lambda t} + \frac{\partial S}{\partial t} = 0,$$

(45)where $C = \left(\frac{2q^2B_0^2 + 8mc^2k}{16mc^2}\right).$

Now, using a change of variables $y = re^{\frac{\lambda t}{2}}$ , we find HJE:

$$+\frac{\partial S}{\partial t} = 0. + Cy^2 \quad \left(\frac{\partial S}{\partial y}\right)^2 \quad \frac{1}{2m}$$

The principal function is

$$S(y,\alpha,t) = \int\sqrt{2m(\alpha - Cy^2)}dy - \alpha t. \qquad (46)$$

Finally, $r = e^{\frac{-\lambda t}{2}}\ \sqrt{\frac{\alpha}{C}}\sin\left((\beta + t)\sqrt{\frac{2C}{m}}\right),$

(47)and $P = \sqrt{2m(\alpha - Cy^2)}\ e^{\frac{\lambda t}{2}}.$

We can quantize the above system by applying the usual rules of canonical quantization. Specifically, we may construct Schrödinger's equation from the Hamiltonian:

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + Cy^2 + \frac{\hbar}{i}\frac{\partial}{\partial t}\right)e^{\frac{iS}{\hbar}} = 0.$$

(48) Using

$$\frac{\partial}{\partial t} e^{\frac{iS(y,t)}{\hbar}} = \frac{i}{\hbar}(-\alpha)\psi; \tag{49}$$

$$\frac{\partial^2}{\partial y^2} e^{\frac{iS(y,t)}{\hbar}} = \frac{-1}{\hbar^2}\psi\left(\frac{\partial S}{\partial y}\right)^2 + \frac{i}{\hbar}\psi\frac{\partial^2 S}{\partial y^2}; \tag{50}$$

so after some algebra and cancellation, taking the semiclassical limit $\hbar \to 0$,we get

$$(\hat{H}_0 + \hat{p}_0)\psi = 0.$$

## 5. Conclusion

This work has focused on quantizing dissipative systems using the WKB approximation. The Hamilton-Jacobi function is used to construct a suitable wave function for such systems.

To test our proposed method, and to get a somewhat deeper understanding, we have examined three examples: the damped harmonic oscillator (together with two "variants": the RLC circuit and a viscous liquid); a system with a variable mass; and a charged particle in a magnetic field. Our formalism may shed further light on such systems as two interacting particles moving in a viscous medium, and the classical radiating electron, among others.

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