DESCRIPTION OF HARMONIC QUASICONFORMAL MAPPINGS

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Abstract

In this paper we will discuss harmonic functions, conformal mappings and quasiconformal mappings and their applications, ect .Which are more flexible than conformal mappings and this make them an easy tool. Conformal mappings degenerate when they are generalized with manny variables, but quasiconformal mappings don’t.

Keywords: Harmonic functions, quasiconformal mappings

Introduction:

Harmonic functions are very important, let see them in detail.

1 Harmonic Functions

If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a point z, then all derivatives of f: $f'(z), f''(z), f'''(z), \dots$ are also analytic in z. [1] From this known fact we will say that all partial derivatives of real functions $u(x,y)$ and $v(x,y)$ are continuous in z. From the fact of continuity of partial derivatives we know that the mixed partial derivatives of second order are equal.

This last fact and the Cauchy- Riemann equations, can be used to demonstrate that is a connection between real and imaginary part of an analytic functions $f(z) = u(x,y) + iv(x,y)$ and the partial derivatives of second order equations

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0.$$

This equation is one of the most famous in applied mathematics, it is known like Laplace’s equations of two variables. The sum $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$ of two partial derivatives of second order is determined from $\nabla^2 \phi$ and is called the laplacian of ϕ. The Laplace’s equation can be writed $\nabla^2 \phi = 0$.

A solution $\phi(x,y)$ of Laplace’s equations in a domen D of the plane is called with a special name.

Definition 1: Harmonic function

A function ϕ of two variables x and y with real values that has partial derivatives of first and second order continuous in a domain D that satisfies the Laplace’s equation is called harmonic on D.

Theorem 1: Harmonic function

Supose that the complex function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D. Than the functions $u(x,y)$ dhe $v(x,y)$ are harmonic in D.

1.1 The harmonic conjugate

If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then the functions u and v are also harmonic in the domain D. Suppose that $u(x,y)$ is a real function which is harmonic in D. If it is possible to find another real function $v(x, y)$ such that u and v satisfies the Cauchy-Riemann's equations in this domain D, then the function $v(x, y)$ is called the harmonic conjugate of $u(x, y)$. If we take the sum of this functions like $u(x, y) + iv(x, y)$ we have a function $f(z) = u(x, y) + iv(x, y)$, which is analytic in the domain D.

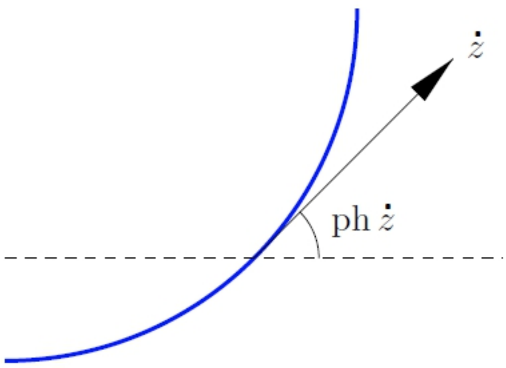
Although most harmonic functions have harmonic conjugates, unfortunately this is not always the case. Interestingly, the existence or non-existence of a harmonic conjugate can depend on the underlying topology of its domain of definition. If the domain is simply connected, and so contains no holes, then one can always find a harmonic conjugate. On non-simply connected domains, there may not exist a single-valued harmonic conjugate to serve as the imaginary part of a complex function f(z).

2 Conformal Mappings

A geometrical property enjoyed by all complex analytic functions is that, at non-critical points, they preserve angles, and therefore define conformal mappings. Conformality makes sense for any inner product space, although in practice one usually deals with Euclidean space equipped with the standard dot product. In the two-dimensional plane, we can assign a sign to the angle between two vectors, whereas in higher dimensions only the absolute value of the angle can be consistently defined.

Definition 2.

A function $g: R^n \to R^n$ is called conformal if it preserves angles. But what does it mean to "preserve angles"? In the Euclidean norm, the angle between two vectors is defined by their dot product. However, most analytic maps are nonlinear, and so will not map vectors to vectors since they will typically map straight lines to curves. However, if we interpret "angle" to mean the angle between two curves , as illustrated in



then we can make sense of the conformality requirement. Thus, in order to realize complex functions as conformal maps, we first need to understand their effect on curves.

In general, a curve C ∈ C in the complex plane is parametrized by a complex-valued Function $z(t) = x(t) + i\, y(t)$, $a \le t \le b$, that depends on a real parameter t. Note that there is no essential difference between a complex curve and a real plane curve; we have merely switched from vector notation $\mathbf{x}(t) = (x(t), y(t))$ to complex notation $z(t) = x(t) + i\, y(t)$.

If $\zeta = g(z)$ is an analytic function and $g'(z) \neq 0$, then g defines a conformal map. One of the most useful consequences stems from the elementary observation that the composition of two complex functions is also a complex function. We re-interpret this operation as a complex change of variables, producing a conformal mapping that preserves angles. A conformal mapping, also called a conformal map, conformal transformation, angle-preserving transformation, or biholomorphic map, is a transformation w = f(z) that preserves local angles. An analytic function is conformal at any point where it has a nonzero derivative. Conversely, any conformal mapping of a complex variable which has continuous partial derivatives is analytic. Conformal mapping is very important in complex analysis, as well as in many areas of physics and engineering.

A mapping that preserves the magnitude of angles, but not their orientation is called an isogonal mapping [2]. Conformal mappings can be effectively used for constructing solutions to the Laplace equation on complicated planar domains that appear in a wide range of physical problems, such as fluid flow, aerodynamics, thermomechanics, electrostatics, and elasticity. [3].

Quasiconformal mappings are generalizations of conformal mappings. [4] They can be considered not only on Riemann surfaces, but also on Riemannian manifolds in all dimensions, and even on arbitrary metric spaces.The importance of quasiconformal mappings in complex analysis was realized by Ahlfors and Teichmüller in the 1930s. Ahlfors used quasiconformalmappings in his geometric approach to Nevanlinna's value distribution theory. Teichmüller used quasiconformal mappings to measure a distance between two conformally inequivalent compact Riemann surfaces, starting what is now called Teichmüller theory.

There are three main definitions for quasiconformal mappings in Euclidean spaces: metric, geometric, and analytic. We give the metric definition,which is the easiest to state and which makes sense in arbitrary metric spaces. It describes the property that"infinitesimal balls are transformed to infinitesimal ellipsoids of bounded eccentricity".

Let $f : X \to Y$ be a homeomorphism between two metric spaces. For $x \in X$ and $r > 0$ let

$$L_f\ (x, r) = \sup\{|f(x) - f(y)| : |x - y| \le r\}$$

and

$$l_f\ (x, r) = \inf\{|f(x) - f(y)| : |x - y| \ge r\}\ .$$

(Here and later we use the Polish notation $|a - b|$ for the distance in any metric space.) The ratio $H_f\ (x, r) = L_f\ (x, r) / l_f\ (x, r)$ measures the eccentricity of the image of the ball $B(x, r)$ under f. We say that f is H-quasiconformal, $H \ge 1$, if

$$\lim_{r \to 0} \sup H_f\ (x, r) \le H \text{ for every } x \in X.$$

Homeomorphisms that are 1-quasiconformal between domains in $R^2 = C$ are precisely the (complex analytic) conformal or anticonformal mappings, by a theorem of Menshov from 1937. Homeomorphisms that are 1-quasiconformal between domains in R^n, $n \ge 3$, are precisely the Möbius transformations, or compositions of inversions on spheres in the one point compactification $R^n \cup \{\infty\}$, by the generalized Liouville theorem proved by Gehring and Reshetnyak in the 1960s. On the other hand, every diffeomorphism $f : R \to R$ is 1-quasiconformal according to the metric definition, as is every homeomorphism between discrete spaces. Surely not all such mappings deserve to be called quasiconformal. [5]

Conclusion:

1. The most common reason is that quasiconformal mappings are the generalization of conformal mappings.
2. We can see that manny theorems of conformal mappings use only quasiconformality.
3. Quasiconformal mappings are more flexible than conformal mappings and this make them an easy tool.
4. Quasiconformal mappings play an important role in some elliptic partial differential equations.
5. The extremal problems of quasiconformal mappings takes to analytic functions related with regions or Riemann's surfaces. This was due to Teichmuller.
6. The problems of moduli were resolved with the help of quasiconformal mappings.
7. Conformal mappings degenérate when they are generalized with manny variables, but quasiconformal mappings don't.

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