PATH INTEGRAL QUANTIZATION OF LAGRANGIANS WITH LINEAR ACCELERATIONS

Eyad H. Hasan
Applied Physics Department, Tafila Technical University, Tafila, Jordan

Abstract

The Lagrangian with linear acceleration can be considered as a model of singular system. The constrained Hamiltonian systems with linear acceleration are investigated by using the Hamilton-Jacobi method. The Hamilton-Jacobi function is constructed by applying the integrability condition on this function to obtain the path integral quantization. It is shown that the equations of motion can be obtained from the action integral.

Keywords: Path Integral, Hamilton-Jacobi, Linear Acceleration

1. Introduction

The efforts to quantize systems with constraints started with the work of Dirac (Dirac, 1950, 1964) who first set up a formalism for treating singular systems. In Dirac's canonical quantization method, the Poisson brackets of classical mechanics are replaced with quantum commutators.

The path integral quantization of constrained systems has been investigated using the canonical method (Rabei, 2000; Muslih, 2001, 2002). In this method the equation of motion are obtained as total differential equations and the set of Hamilton-Jacobi partial differential equations is determined.

Recently, the quantization of constrained systems has been studied using the WKB approximation (Rabei *et al*., 2002; Rabei *et al*., 2005; Hasan *et al*., 2004). The set of Hamilton-Jacobi partial differential equations for these systems has been determined using the canonical method. The Hamilton-Jacobi function has been obtained by solving these equations. By calculating the Hamilton-Jacobi function and constructing the wave function, the quantization has been carried out using this approximation.

Some authors (Rabei *et al*., 2003) have investigated singular Lagrangians with linear velocities by using the Hamilton-Jacobi method and obtained the integrable action directly without considering the total variation of constraints. We wish to extend the model for second-order linear Lagrangian.

This paper is organized as follow. In Sec. 2, the canonical path integral formalism for second-order Lagrangians is reviewed briefly. In Sec. 3, a new model of singular Lagrangian with linear acceleration is proposed. In Sec. 4, several illustrative examples are examined. The work closes with some concluding remarks in Sec. 5.

2. The canonical path integral formalism for second-order Lagrangians

The Lagrangian formulation of second-order theories requires the configuration space formed by N generalized coordinates q_i, N generalized velocities $\dot{q}_i$, and N generalized accelerations $\ddot{q}_i$:

$$L \equiv L(q_i, \dot{q}_i, \ddot{q}_i), \qquad i = 1, \dots, N . \tag{2.1}$$

The corresponding Euler-Lagrangian equations of motion are obtained from

$$S = \int L(q_i, \dot{q}_i, \ddot{q}_i) dt$$

using the Hamilton principle:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) = 0 . \tag{2.2}$$

This is a system of N differential equations of fourth order; so we need 4N initial conditions to solve it.

The Hamiltonian formulation for second-order derivatives, first developed by (Ostrogradski, 1850), treats q_i and $\dot{q}_i$ as coordinates. The transformation from the Lagrangian to the Hamiltonian approach is achieved by introducing the generalized momenta p_i, π_i conjugate to the generalized coordinates q_i, $\dot{q}_i$, respectively:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right); \tag{2.3}$$

$$\pi_i = \frac{\partial L}{\partial \ddot{q}_i}, \tag{2.4}$$

then writing the accelerations $\ddot{q}_i$ as functions of the coordinates q_i and velocities $\dot{q}_i$ as well as of the momenta p_i and π_i $\left[\ddot{q}_i = f(q_i, \dot{q}_i, p_i, \pi_i)\right]$. The phase space will then be spanned by the canonical variables (q_i, p_i) and $(\dot{q}_i, \pi_i)$.

I ntroducing the canonical Hamiltonian

$$H_0 \equiv p_i \dot{q}_i + \pi_i \ddot{q}_i \big|_{\ddot{q}_i = f_i} - L\big|_{\ddot{q}_i = f_i}, \tag{2.5}$$

one can write the equations of motion of any function g of the canonical variables as

$$\dot{g} = \{g, H_0\}. \tag{2.6}$$

where $\{,\}$ is the Poisson bracket defined as

$$\{A, B\} = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} + \frac{\partial A}{\partial \dot{q}_i}\frac{\partial B}{\partial \pi_i} - \frac{\partial A}{\partial \pi_i}\frac{\partial B}{\partial \dot{q}_i}.$$

Here A and B are functions in the phase space described in terms of the canonical variables $q_i, \dot{q}_i, p_i, \pi_i$, which obey 4N first-order differential equations of motion.

However, this procedure is admissible only if the determinant of the Hessian matrix,

$$H_{ij} \equiv \left(\frac{\partial^2 L}{\partial \ddot{q}_i \partial \ddot{q}_j} \right), \qquad i, j = 1, \ldots, N$$

does not vanish; otherwise it will not be possible to express all the accelerations $\ddot{q}_i$ as functions of the canonical variables, and there will be relations such as

$$\Phi_\alpha(q_i, p_i; \dot{q}_i, \pi_i) = 0, \qquad \alpha = 1, \ldots, m < 2(N-1),$$

connecting the momenta variables. As a consequence, we will not be able to treat the canonical variables as an independent set; instead, we will have to use a formalism specially developed to deal with the interdependent canonical variables, i.e., a formalism for constrained systems (Dirac, 1950, 1964).

Let us consider a Lagrangian $L(q_i, \dot{q}_i, \ddot{q}_i, t)$. One can obtain a completely equivalent Lagrangian by introducing

$$L' = L(q_i, \dot{q}_i, \ddot{q}_i, t) - \frac{dS(q_i, \dot{q}_i, t)}{dt}, \tag{2.7}$$

such that the auxiliary function $S(q_i, \dot{q}_i, t)$ must satisfy

$$\frac{\partial S}{\partial t} = -H_0, \tag{2.8}$$

where H_0 is defined as the usual Hamiltonian:

$$H_0 = p_i \dot{q}_i + \pi_i \ddot{q}_i - L; \tag{2.9}$$

$$p_i = \frac{\partial S}{\partial q_i}; \tag{2.10}$$

$$\pi_i = \frac{\partial S}{\partial \dot{q}_i} . \tag{2.11}$$

These are the fundamental equations of the equivalent Lagrangian method; Eq. (2.8) is the relevant Hamilton-Jacobi partial differential equation.

If the rank of the Hessian matrix

$$\frac{\partial^2 L}{\partial \ddot{q}_i \partial \ddot{q}_j} \tag{2.12}$$

is N-R, R<N, then the generalized momenta conjugate to the generalized coordinates $\dot{q}_i$ are defined as

$$\pi_a = \frac{\partial L}{\partial \ddot{q}_a}, \qquad a = R+1,...,N; \tag{2.13}$$

$$\pi_\alpha = \frac{\partial L}{\partial \ddot{q}_\alpha}, \qquad \alpha = 1,...,R. \tag{2.14}$$

Since the rank of the Hessian matrix is N-R, one can solve Eq. (2.13) to obtain N-R accelerations $\ddot{q}_a$ in terms of q_i, $\dot{q}_i$, π_a and $\ddot{q}_\alpha$ as follows:

$$\ddot{q}_a = w_a\left(q_i, \dot{q}_i, \pi_a, \ddot{q}_\alpha\right). \tag{2.15}$$

Substituting Eq. (2.15) into (2.14), one gets

$$\pi_\alpha = \frac{\partial L}{\partial \ddot{q}_\alpha} \Big|_{\ddot{q}_a = w_a\left(q_i, \dot{q}_i, \pi_a, \ddot{q}_\alpha\right)} = -H_\alpha^\pi\left(q_i, \dot{q}_i, p_a, \pi_a\right). \tag{2.16}$$

We can obtain a similar expression for the momenta p_α:

$$p_\alpha = -H_\alpha^p\left(q_i, \dot{q}_i, p_a, \pi_a\right), \tag{2.17}$$

where

$$p_a = \frac{\partial L}{\partial \dot{q}_a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_a}\right); \tag{2.18a}$$

$$p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_\alpha}\right). \tag{2.18b}$$

Equations (2.16) and (2.17) become

$$H_\alpha'^\pi\left(q_i, \dot{q}_i, p_i, \pi_i\right) = \pi_\alpha + H_\alpha^\pi = 0; \tag{2.19a}$$

$$H_\alpha'^p\left(q_i, \dot{q}_i, p_i, \pi_i\right) = p_\alpha + H_\alpha^p = 0, \tag{2.19b}$$

which are called primary constraints (Dirac, 1950, 1964). These relations indicate that the generalized momenta p_α and π_α are not independent of p_a

and π_a, which is a natural result of the singular nature of the Lagrangian. The Hamiltonian H_0 is then defined as

$$H_0 = p_a\dot{q}_a + \dot{q}_\alpha p_\alpha \Big|_{p_\beta=-H_\beta^p} + \pi_a\ddot{q}_a + \ddot{q}_\alpha\pi_\alpha \Big|_{\pi_\beta=-H_\beta^\pi} - L(q_i, \dot{q}_i, \ddot{q}_\alpha, \ddot{q}_a = w_a);$$
$$\beta = 1,\ldots,R; \qquad a = R+1,\ldots,N. \tag{2.20}$$

Defining the momentum p_0 as

$$p_0 = \frac{\partial S}{\partial t}, \tag{2.21}$$

one can write the corresponding set of HJPDEs as (Pimentel and Teixeira, 1996)

$$H_0' = p_0 + H_0 = p_0 + H_0\left(t, q_\alpha, \dot{q}_\alpha, q_a, \dot{q}_a; p_a = \frac{\partial S}{\partial q_a}; \pi_a = \frac{\partial S}{\partial \dot{q}_a}\right) = 0; \tag{2.22a}$$

$$H_\alpha'^p = p_\alpha + H_\alpha^p = p_\alpha + H_\alpha^p\left(t, q_\alpha, \dot{q}_\alpha, q_a, \dot{q}_a; p_a = \frac{\partial S}{\partial q_a}; \pi_a = \frac{\partial S}{\partial \dot{q}_a}\right) = 0; \tag{2.22b}$$

$$H_\alpha'^\pi = \pi_\alpha + H_\alpha^\pi = \pi_\alpha + H_\alpha^\pi\left(t, q_\alpha, \dot{q}_\alpha, q_a, \dot{q}_a; p_a = \frac{\partial S}{\partial q_a}; \pi_a = \frac{\partial S}{\partial \dot{q}_a}\right) = 0. \tag{2.22c}$$

The equations of motion are written as total differential equations in many variables as follows (Pimentel and Teixeira, 1996):

$$dq_a = \frac{\partial H_0'}{\partial p_a}dt + \frac{\partial H_\alpha'^p}{\partial p_a}dq_\alpha + \frac{\partial H_\alpha'^\pi}{\partial p_a}d\dot{q}_\alpha; \tag{2.23a}$$

$$d\dot{q}_a = \frac{\partial H_0'}{\partial \pi_a}dt + \frac{\partial H_\alpha'^p}{\partial \pi_a}dq_\alpha + \frac{\partial H_\alpha'^\pi}{\partial \pi_a}d\dot{q}_\alpha; \tag{2.23b}$$

$$dp_i = -\frac{\partial H_0'}{\partial q_i}dt - \frac{\partial H_\alpha'^p}{\partial q_i}dq_\alpha - \frac{\partial H_\alpha'^\pi}{\partial q_i}d\dot{q}_\alpha; \tag{2.23c}$$

$$d\pi_i = -\frac{\partial H_0'}{\partial \dot{q}_i}dt - \frac{\partial H_\alpha'^p}{\partial \dot{q}_i}dq_\alpha - \frac{\partial H_\alpha'^\pi}{\partial \dot{q}_i}d\dot{q}_\alpha; \tag{2.23d}$$

$$\begin{aligned} dZ &= (-H_\circ + p_a\frac{\partial H_0'}{\partial p_a} + \pi_a\frac{\partial H_\circ'}{\partial \pi_a})dt \\ &+(-H_\alpha^p + p_a\frac{\partial H_\alpha'^{\,p}}{\partial p_a} + \pi_a\frac{\partial H_\alpha'^p}{\partial \pi_a})dq_\alpha \\ &+(-H_\alpha^\pi + p_a\frac{\partial H_\alpha'^{\,\pi}}{\partial p_a} + \pi_a\frac{\partial H_\alpha'^\pi}{\partial \pi_a})d\dot{q}_\alpha. \end{aligned} \tag{2.23e}$$

where $Z = S(q_\alpha, \dot{q}_\alpha, q_a, \dot{q}_a)$. We note that the existence of constraints reduces the number of the equations of motion.

Here $q_0 = t$. The set of Eqs. (2.23) is integrable (Muslih and Guler, 1998) if and only if

$$dH'_0 \equiv 0; \tag{2.24a}$$

$$dH'^{p}_{\alpha} \equiv 0; \tag{2.24b}$$

$$dH'^{\pi}_{\alpha} \equiv 0. \tag{2.24c}$$

These conditions are identically satisfied or they lead to new constraints.

Besides the canonical action integral is obtained in terms of the canonical coordinates. H'_0 , H'^{p}_{α} and H'^{π}_{α} can be interpreted as the infinitesimal generators of canonical transformations given by parameters t, q_α and $\dot{q}_\alpha$ respectively. In this case, the path integral representation may be written as (Guler, Y. 1992; Muslih and Guler, 1998; Rabei, 2000; Muslih, 2001, 2002)

$$\langle q_a, \dot{q}_a, q_\alpha, \dot{q}_\alpha \,|\, q'_a, \dot{q}'_a, q'_\alpha, \dot{q}'_\alpha \rangle =$$

$$\int \prod_{a=1}^{R} dq_a dp_a \prod_{a=1}^{R} d\dot{q}_a d\pi_a \times \exp i\{ \int_{q_\alpha, \dot{q}_\alpha}^{q'_\alpha, \dot{q}'_\alpha} (-H_\circ + p_a \frac{\partial H'_\circ}{\partial p_a} + \pi_a \frac{\partial H'_\circ}{\partial \pi_a}) dt$$

$$+(-H^{p}_{\alpha} + p_a \frac{\partial H'^{\ p}_{\alpha}}{\partial p_a} + \pi_a \frac{\partial H'^{p}_{\alpha}}{\partial \pi_a}) dq_\alpha$$

$$+(-H^{\pi}_{\alpha} + p_a \frac{\partial H'^{\ \pi}_{\alpha}}{\partial p_a} + \pi_a \frac{\partial H'^{\pi}_{\alpha}}{\partial \pi_a}) d\dot{q}_\alpha \} \tag{2.25}$$

3. The Model

The general form of a second-order linear Lagrangian is

$$L(q_i, \dot{q}_i, \ddot{q}_i) = a_i(q_j, \dot{q}_j)\ddot{q}_i - V(q_j, \dot{q}_j) \tag{3.1}$$

The associated Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) = 0 \tag{3.2}$$

have at most order three. If $a_i(q,\dot{q}) = a_i(q)$,then the associated Euler-Lagrange equations have at most order two. Let $V(q,\dot{q}) = V(q)$, in this case the general form of a second-order linear Lagrangian becomes

$$L(q_i, \dot{q}_i, \ddot{q}_i) = a_i(q_j)\ddot{q}_i - V(q_j) \tag{3.3}$$

The transformation from the Lagrangian to the Hamiltonian approach is achieved by introducing the generalized momenta p_i, π_i conjugate to the generalized coordinates q_i, $\dot{q}_i$, respectively:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right);$$

$$p_i = -\frac{da_i}{dt} = b_i(\dot{q}_j) = -H_i^p; \tag{3.4}$$

$$\pi_i = \frac{\partial L}{\partial \ddot{q}_i} = a_i(q_j) = -H_i^\pi. \tag{3.5}$$

Equations (4) and (5) become

$$H_i'^\pi\left(q_i, \dot{q}_i, p_i, \pi_i\right) = \pi_i + H_i^\pi = 0;$$

$$H_i'^\pi\left(q_i, \dot{q}_i, p_i, \pi_i\right) = \pi_i - a_i = 0; \tag{3.6}$$

$$H_i'^p\left(q_i, \dot{q}_i, p_i, \pi_i\right) = p_i + H_i^p = 0;$$

$$H_i'^p\left(q_i, \dot{q}_i, p_i, \pi_i\right) = p_i - b_i = 0. \tag{3.7}$$

which are called primary constraints.

The canonical Hamiltonian H_0 is given by:

$$H_0 = p_i\dot{q}_i + \pi_i\ddot{q}_i - L = b_i(\dot{q}_j)\dot{q}_i + V(q_j) \tag{3.8}$$

The corresponding HJPDEs

$$H'_\circ = \frac{\partial S}{\partial t} + b_i\dot{q}_i + V(q_j) = 0;$$

$$H'^\pi_\circ = \frac{\partial S}{\partial \dot{q}} - a_i = 0;$$

$$H_i'^p = \frac{\partial S}{\partial q} - b_i = 0.$$

The equations of motion are obtained as total differential equations follows:

$$dq_i = \frac{\partial H_0'}{\partial p_i}dt + \frac{\partial H_j'^p}{\partial p_i}dq_j + \frac{\partial H_j'^\pi}{\partial p_i}d\dot{q}_j = dq_j; \tag{3.9}$$

$$d\dot{q}_i = \frac{\partial H_0'}{\partial \pi_i}dt + \frac{\partial H_j'^p}{\partial \pi_i}dq_j + \frac{\partial H_j'^\pi}{\partial \pi_i}d\dot{q}_j = d\dot{q}_j; \tag{3.10}$$

$$dp_i = -\frac{\partial H_0'}{\partial q_i}dt - \frac{\partial H_j'^p}{\partial q_i}dq_j - \frac{\partial H_j'^\pi}{\partial q_i}d\dot{q}_j = -\frac{\partial V}{\partial q_i}dt + \frac{\partial a_j}{\partial q_i}d\dot{q}_j; \tag{3.11}$$

$$d\pi_i = -\frac{\partial H_0'}{\partial \dot{q}_i}dt - \frac{\partial H_\alpha'^p}{\partial \dot{q}_i}dq_\alpha - \frac{\partial H_\alpha'^\pi}{\partial \dot{q}_i}d\dot{q}_\alpha = -b_i(\dot{q}_j)dt. \tag{3.12}$$

To check whether the set of equations (9-12) are integrable or not , let us consider the total variation of (6) and (7). In fact

$$dH_i'^{\pi} = d\pi_i - da_i = 0$$

$$= -b_i(\dot{q}_j)dt - da_i(q_j); \qquad (3.13)$$

$$dH_i'^{p} = dp_i - db_i = 0$$

$$= -\frac{\partial V}{\partial q_i}dt + \frac{\partial a_j}{\partial q_i}d\dot{q}_j - db_i(\dot{q}_j); \qquad (3.14)$$

So, we have

$$\frac{\partial b_i(\dot{q})}{\partial \dot{q}_j}d\dot{q}_j - \frac{\partial a_j(q)}{\partial q_i}d\dot{q}_j = -\frac{\partial V}{\partial q_i}dt \;; \qquad (3.15)$$

which is equivalent to

$$\frac{\partial a_i(q)}{\partial q_j}d\dot{q}_j + \frac{\partial a_j(q)}{\partial q_i}d\dot{q}_j = \frac{\partial V}{\partial q_i}dt \;; \qquad (3.16)$$

or $$\ddot{q}_j = f_{ij}^{-1}\frac{\partial V(q)}{\partial q_i};$$

where the symmetric matrix f_{ij} is given by

$$f_{ij} = \frac{\partial a_i(q)}{\partial q_j} + \frac{\partial a_j(q)}{\partial q_i}. \qquad (3.17)$$

The total derivative of the Hamilton-Jacobi function can be obtained as:

$$dS = \frac{\partial S}{\partial q_i}dq_i + \frac{\partial S}{\partial \dot{q}_i}d\dot{q}_i + \frac{\partial S}{\partial t}dt. \qquad (3.18)$$

Using the above HJPDEs, we get

$$dS = a_i d\dot{q}_i - Vdt,$$

Which can integrated to give

$$S = \int a_i d\dot{q}_i - \int Vdt, \qquad (3.19)$$

Now, using the fact that

$$\int d(a_i\dot{q}_i) = a_i\dot{q}_i = \int a_i d\dot{q}_i + \int \dot{q}_i da_i,$$

The above Hamilton-Jacobi function Eq. (3.16) reduces to

$$S = \frac{1}{2}[a_i\dot{q}_i + \int a_i d\dot{q}_i - \int \dot{q}_j da_j] - \int Vdt \;, \qquad (3.20)$$

By some rearragment, Eq. 3.17 becomes

$$S = \frac{1}{2} a_i \dot{q}_i - \frac{1}{2} \int [\dot{q}_j da_j - a_i d\dot{q}_i + 2V dt], \tag{3.21}$$

And using the fact that

$$\frac{d}{dt}(q_j da_j) = -q_j db_j + \dot{q}_j da_j,$$

$$S = \frac{1}{2} a_i \dot{q}_i - \frac{1}{2} \int \frac{d}{dt}(q_j da_j) - \frac{1}{2} \int [q_j db_j - a_i d\dot{q}_i + 2V dt]$$

$$S = \frac{1}{2} a_i \dot{q}_i - \frac{1}{2} \frac{d}{dt} \int q_j da_j - \frac{1}{2} \int [q_j db_j - a_i d\dot{q}_i + 2V dt], \tag{3.22}$$

Assuming that the function $a_i(q)$ and $V(q)$ satisfy the following conditions

$$q_j \frac{\partial a_i}{\partial q_j} = a_i, \qquad \frac{\partial V}{\partial q_j} q_j = 2V$$

Eq. (3.22) becomes

$$S = \frac{1}{2} a_i \dot{q}_i - \frac{1}{2} \frac{d}{dt} \int q_j da_j - \frac{1}{2} \int q_j [db_j - \frac{\partial a_i}{\partial q_j} d\dot{q}_i + \frac{\partial V}{\partial q_j} dt]. \tag{3.23}$$

However, in order for S to be an integrable function, the terms in the brackets must be zero, i.e.

$$db_j - \frac{\partial a_i}{\partial q_j} d\dot{q}_i + \frac{\partial V}{\partial q_j} dt = 0. \tag{3.24}$$

In fact, this equation represents the equation of motion for the coordinates q_j.

To obtain the path integral quantization for the singular Lagrangian (3.3), we have two different cases,

Case1: If the inverse of the matrix f_{ij} exist, then we can solve all the dynamics q_i. In this case the path integral Ψ is given by

$$\Psi = \int \prod_{i=1}^{n} dq_i d\dot{q}_i \exp i\{\frac{1}{2} a_i \dot{q}_i - \frac{1}{2} \frac{d}{dt} \int q_i da_i + c\}. \tag{3.25}$$

Case2: If the rank of the matrix f_{ij} is n-R, then we can solve the dynamics q_a in terms of independent parameters $(t, q_\alpha, \dot{q}_\alpha)$, $\alpha = 1, 2, ..., R$. In this case the path integral Ψ is given by

$$\Psi = \int \prod_{a=1}^{n-R} dq_a d\dot{q}_a \exp i\{\frac{1}{2} a_i \dot{q}_i - \frac{1}{2} \frac{d}{dt} \int q_i da_i + c\}, \quad i = 1, 2, ..., n \tag{3.26}$$

4. Examples

4.1 As a first example, we consider the following linear (singular) Lagrangian (Goldstein, 1980) .

$$L = -\frac{1}{2} mq\ddot{q} - \frac{1}{2} kq^2 \tag{4.1}$$

where the potential V is given by

$$V = \frac{1}{2} kq^2$$

Here the function a is

$$a = -\frac{1}{2} mq$$

Using (3.4) and (3.5), the generalized momenta corresponding to this Lagrangian are:

$$p = \frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right) = \frac{1}{2} m\dot{q} = -H^{p}; \tag{4.2a}$$

$$\pi = \frac{\partial L}{\partial \ddot{q}} = -\frac{1}{2} mq = -H^{\pi} . \tag{4.2b}$$

By (3.6) and (3.7) the primary constraints are given as

$$H'^{\pi} = \pi + \frac{1}{2} mq; \tag{4.3a}$$

$$H'^{p} = p - \frac{1}{2} m\dot{q} . \tag{4.3b}$$

Equation (3.8) gives the canonical Hamiltonian H_0 as

$$H_{\circ} = \frac{1}{2} m\dot{q}^2 + \frac{1}{2} kq^2 \tag{4.4}$$

Now using (3.11) and (3.12), the equations of motion read as

$$dp = -kqdt - \frac{1}{2} md\dot{q}; \tag{4.5a}$$

$$d\pi = \frac{1}{2} m\dot{q}dt . \tag{4.5b}$$

Making use of (3.24), we can obtain the equation of motion for q

$$\frac{1}{2} md\dot{q} + \frac{1}{2} md\dot{q} + kqdt = 0, \tag{4.6}$$

This equation can be written as

$$\ddot{q} + \frac{k}{m} q = 0 . \tag{4.7}$$

This is interesting because this equation of motion is just Hook's law. It's interesting to notice that this equation is familiar for a simple harmonic oscillator, which has the following solution

$q = A\cos\omega t + B\sin\omega t$, where $\omega = \sqrt{\frac{k}{m}}$.

Using Eq. (3.23), we find the integrable action function as

$$S = c \tag{4.8}$$

Making use of equation (3.25) and (4.8), the path integral is given by

$$\Psi = \int dq d\dot{q} \exp ic . \tag{4.9}$$

4.2. As a second example consider the following singular Lagrangian:

$$L = q_1\ddot{q}_1 + q_2\ddot{q}_2 - \frac{1}{2}(q_2^2 + q_2^2), \tag{4.10}$$

where the potential V is given by

$$V = \frac{1}{2}(q_1^2 + q_2^2)$$

Here the functions a_1 and a_2 are

$$a_1 = q_1, \qquad a_2 = q_2$$

Using (3.4) and (3.5), the generalized momenta corresponding to this Lagrangian are:

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_1}\right) = -\dot{q}_1 = -H_1^p; \tag{4.11a}$$

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_2}\right) = -\dot{q}_2 = -H_2^p; \tag{4.11b}$$

$$\pi_1 = \frac{\partial L}{\partial \ddot{q}_1} = q_1 = -H_1^\pi; \tag{4.11c}$$

$$\pi_2 = \frac{\partial L}{\partial \ddot{q}_2} = q_2 = -H_2^\pi. \tag{4.11d}$$

By (3.6) and (3.7) the primary constraints are given as

$$H_1'^\pi = \pi_1 - q_1; \tag{4.12a}$$

$$H_2'^\pi = \pi_2 - q_2; \tag{4.12b}$$

$$H_1'^p = p_1 + \dot{q}_1; \tag{4.12c}$$

$$H_2'^p = p_2 + \dot{q}_2. \tag{4.12d}$$

Equation (3.8) gives the canonical Hamiltonian H_0 as

$$H_{\circ} = -\dot{q}_1^2 - \dot{q}_2^2 + \frac{1}{2}(q_1^2 + q_2^2)\,. \qquad (4.13)$$

Now using (3.11) and (3.12), the equations of motion read as

$$dp_1 = -q_1 dt + d\dot{q}_1\,; \qquad (4.14a)$$

$$dp_2 = -q_2 dt + d\dot{q}_2\,; \qquad (4.14b)$$

$$d\pi_1 = \dot{q}_1 dt\,; \qquad (4.14c)$$

$$d\pi_2 = \dot{q}_2 dt\,. \qquad (4.14d)$$

The matrix f_{ij} defined in (3.17) is given by

$$f_{ij} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad (4.15)$$

Making use of (3.24), we can obtain the equation of motion for q_1 and q_2

$$-d\dot{q}_1 + q_1 dt - d\dot{q}_1 = 0\,, \qquad (4.16a)$$

$$-d\dot{q}_2 + q_2 dt - d\dot{q}_2 = 0\,. \qquad (4.16b)$$

These equations can be written as

$$2\ddot{q}_1 - q_1 = 0\,; \qquad (4.17a)$$

$$2\ddot{q}_2 - q_2 = 0\,. \qquad (4.17b)$$

Which have the following solutions

$$q_1 = Ae^{t/\sqrt{2}} + Be^{-t/\sqrt{2}}\,; \qquad (4.18a)$$

$$q_2 = Ae^{t/\sqrt{2}} + Be^{-t/\sqrt{2}}\,. \qquad (4.18b)$$

Using Eq. (3.23), we find the integrable action function as

$$S = c\,. \qquad (4.19)$$

Making use of equation (3.25) and (4.19), the path integral is given by

$$\Psi = \int dq_1 dq_2 d\dot{q}_1 d\dot{q}_2 \exp ic\,. \qquad (4.20)$$

4.3. As a third example consider the following singular Lagrangian:

$$L = q_2\ddot{q}_1 - q_1\ddot{q}_2 + q_3\ddot{q}_3 - \frac{1}{2}(q_2^2 + q_2^2 + q_3^2)\,, \qquad (4.21)$$

where the potential V is given by

$$V = \frac{1}{2}(q_1^2 + q_2^2 + q_3^3)$$

Here the functions a_1, a_2 and a_3 are

$$a_1 = q_2\,, \qquad a_2 = -q_1\,, \qquad a_3 = q_3$$

Using (3.4) and (3.5), the generalized momenta corresponding to this Lagrangian are:

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_1}\right) = -\dot{q}_2 = -H_1^p ; \tag{4.22a}$$

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_2}\right) = \dot{q}_1 = -H_2^p ; \tag{4.22b}$$

$$p_3 = \frac{\partial L}{\partial \dot{q}_3} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_3}\right) = -\dot{q}_3 = -H_2^p ; \tag{4.22c}$$

$$\pi_1 = \frac{\partial L}{\partial \ddot{q}_1} = q_2 = -H_1^\pi ; \tag{4.22d}$$

$$\pi_2 = \frac{\partial L}{\partial \ddot{q}_2} = -q_1 = -H_2^\pi ; \tag{4.22e}$$

$$\pi_3 = \frac{\partial L}{\partial \ddot{q}_3} = q_3 = -H_3^\pi . \tag{4.22f}$$

By (3.6) and (3.7) the primary constraints are given as

$$H_1'^\pi = \pi_1 - q_2 ; \tag{4.23a}$$

$$H_2'^\pi = \pi_2 + q_1 ; \tag{4.23b}$$

$$H_3'^\pi = \pi_3 - q_3 ; \tag{4.23c}$$

$$H_1'^p = p_1 + q_2 ; \tag{4.23d}$$

$$H_2'^p = p_2 - \dot{q}_1 ; \tag{4.23e}$$

$$H_3'^p = p_3 + \dot{q}_3 . \tag{4.23f}$$

Equation (3.8) gives the canonical Hamiltonian H_0 as

$$H_0 = -\dot{q}_3^2 + \frac{1}{2}(q_1^2 + q_2^2 + q_3^2) . \tag{4.24}$$

Now using (3.11) and (3.12), the equations of motion read as

$$dp_1 = -q_1 dt - d\dot{q}_2 ; \tag{4.25a}$$

$$dp_2 = -q_2 dt + d\dot{q}_1 ; \tag{4.25b}$$

$$dp_3 = -q_3 dt + d\dot{q}_3 ; \tag{4.25c}$$

$$d\pi_1 = \dot{q}_2 dt ; \tag{4.25d}$$

$$d\pi_2 = -\dot{q}_1 dt ; \tag{4.25e}$$

$$d\pi_3 = \dot{q}_3 dt . \tag{4.25f}$$

The matrix f_{ij} defined in (3.17) is given by

$$f_{ij} = \begin{pmatrix} 0 0 0 \\ 0 0 0 \\ 0 0 2 \end{pmatrix} \tag{4.26}$$

Making use of (3.24), we can obtain the equation of motion for q_3

$$-d\dot{q}_3 + q_3 dt - d\dot{q}_3 = 0 \ . \tag{4.27}$$

This equation can be written as

$$2\ddot{q}_3 - q_3 = 0, \tag{4.28}$$

which have the following solution

$$q_3 = Ae^{t/\sqrt{2}} + Be^{-t/\sqrt{2}} . \tag{4.29}$$

Using Eq. (3.23), we find the integrable action function as

$$S = \frac{1}{2} q_2 \dot{q}_1 - \frac{1}{2} q_1 \dot{q}_2 + c . \tag{4.30}$$

Making use of equation (3.26) and (4.30), the path integral is given by

$$\Psi = \int dq_3 d\dot{q}_3 \exp i[\frac{1}{2} q_2 \dot{q}_1 - \frac{1}{2} q_1 \dot{q}_2 + c] . \tag{4.31}$$

5. Conclusion

In this work, we have investigated the singular Lagrangians with linear accelerations using the Hamilton-Jacobi method and obtained the integrable action directly without considering the total variation of constrained. In other words, it has been shown that the total derivative of the Hamilton-Jacobi function is constructed using the HJPDEs. In order to show that this function is integrable, some conditions must be satisfied. It is shown that these conditions represent the equations of motion which are equivalent to both the consistency conditions of Dirac method and the canonical method. Also it is shown that by calculating the integrable action and constructing the wave function, the quantization has been carried out using the path integral.

References:

Dirac, P. A. M. 1950. Generalized Hamiltonian Dynamics. *Canadian Journal of Mathematical Physics*, 2, 129-148.

Dirac, P. A. M. 1964. *Lectures on Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University, New York.

Rabei, E. M. 2000. On the Quantization of Constrained Systems Using Path Integral Techniques. Il Nuovo Cimento B, 115(10), 1159-1165.

Muslih, S. I. 2001. Path Integral Formulation of Constrained Systems With Singular Higher-Order Lagrangians. Hadronic Journal, 24, 713-721.

Muslih, S. I. 2002. Quantization of Singular Systems with Second-Order Lagrangians. Modern Physics Letters A, 17, 2383-2391.

Rabei, E. M., Nawafleh, K. I., and Ghassib, H. B. 2002. Quantization of Constrained Systems Using the WKB Approximation. Physical Review A, 66, 024101-024106.

Rabei, E. M., Hasan, E. H., Ghassib, H. B and Muslih, S 2005. Quantization of Second-Order Constrained Lagrangian Systems using the WKB Approximation, *International Journal of Geometric Methods in Modern Physics*, 2 (3), 485-504.

Hasan, E. H ., Rabei, E. M., and Ghassib, H. B 2004. Quantization of Higher-Order Constrained Lagrangian Systems Using the WKB Approximation. *International Journal of Theoretical Physics*,43 (4), 1073-1096.

Rabei, E. M., Nawafleh K. I., Y. S. Abdelrahman and H.Y R. Omari 2003. Hamilton-Jacobi treatment of lagrangians with linear velocities, *Modern Physics Letters A*, 18 (23), 1591-1596.

M. Ostrogradskii, *Mem. Ac. St. Peterspourg* 1 (1850) 385.

Pimentel, R.G. and Teixeira 1996. Hamilton-Jacobi Formulation for Singular Systems with Second-Order Lagrangians. *Il Nuovo Ciemento B*, 111, 841-854.

Muslih S. I and Guler Y. Is gauge fixing of constrained systems necessary. Nuovo Cimento B, 113 (1998) 277-290

Guler, Y. 1992. Integration of Singular Systems. *Il Nuovo Cimento B*, 107 (12),1389-1395.

Guler, Y. 1992b. Canonical Formulation of Singular Systems. *Il Nuovo Cimento B*, 107 (10), 1143-1149.

Goldstein, H. 1980. *Classical Mechanics*, 2nd edition, Addison-Wesley, Reading- Massachusetts.