MATHEMATICS AND PHYSICS
ESTIMATES OF SLOW CONTROLS

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Abstract

We consider a linear control system with the origin as target and study the behavior of the minimum norm control as time duration goes to infinity.

Keywords: control systems, minimum time, minimum norm

Introduction

Consider the abstract control system

$$y' = Ay + Bu \tag{3.1}$$

where A generates a continuous semigroup in a Banach space X and B is linear and bounded from a Banach space U to X. This setting is very general and covers most systems encountered in applications: distributed control systems, point control systems, neutral functional differential equations.

The solution of equation (3.1), which satisfies the initial condition y(0)=x, is given by

$$y(t,s,u) = S(t)x + \int_0^t S(t-s)\, Bu(s)\, ds.$$

Here S(t) is the linear semigroup generated by A. The minimum energy (norm control) to bring x to zero in time t is denoted by E(t,x).

Suppose that the system (3.1) is null-controllable on some time interval t, i.e., for each x in X there exists u(.) such that y(t, x, u)=0. Obviously, if a state x can be steered to zero in some time, then it can be steered to zero in any larger time and it is expected that, as the time grows, the control energy needed to transfer x to zero to become smaller. In this paper, we study the behavior of the minimum energy to transfer x to zero when the transferable time tends to infinity..

The reverse case, of fast controls (when t is small), was treated in the finite dimensional setting in [10] for L^2 -controls and in [11] for L^p -controls, with p in the interval $(1, \infty)$ and there were given asymptotics of the minimum norm control as t tends to zero. In infinite dimensional case, this problem was studied in [12] for coupled systems of partial differential equations.

In [13], for large t, there are given estimates for the control cost, for an abstract parabolic equations of form (3.1) (A being a generator of a contraction semigroup) in Hilbert spaces, with L^2 -controls.

In [8], W. Krabs considered a control systems described by an abstract wave equation of the form y" =Ay +u in a Hilbert space and studied the null controllability for every positive time with L^∞-controls. He showed that for every positive t, the null controllability is possible by a minimum norm control which is unique on [0, t] and satisfies a bang-bang principle. Moreover, he proved that the norm control tends to zero when t tends to infinity. Similar results have been proved in [7] for L^2 -controls. We mention also the paper of S. Ivanov [6] where it is studied the rate for the minimal energy to tend to 0 as t tends to infinity, for second order control systems with L^2 and L^1 -controls. The constrained null contollability of the system (3.1) was proved in [2] for contaction semigroups. Estimates for fast controls

The above result says that a function r(t) satisfying (3.3) on some interval $[0, T_0]$ can be extended on the whole interval $[0, \infty)$ again satisfying (3.3). Moreover, we are interested in getting a continuous (eventually, strictly increasing) function satisfying (3.3) on $[T_{0,} \infty)$. To this end, recall that, considering a suitable norm on X, for some ω we have $|S(t)| \leq \exp(\omega t)$ for any $t \geq 0$. Let us define the function r(t) on $[T_0, \infty)$ by $r(t)= (r(T_0)(\sum \exp (-\omega\, kT_0-\omega q) + r(q))$, $k=0,1, .. n-1$, $t= q+nT_0$ with q in $(0, T_0]$. It is clear that if r(t) is continbuous on $[0, T_0]$, then its continuation defined as above is also continuous for any $t> T_0$, and satisfies the inclusion (3.3).

were also obtained in [3] in the finite dimensional setting as well as in some infinite dimensional cases.

For any positive t, let us define the operator V(t) by

$V(t)u = \int_0^t S(t-s)\, Bu(s)\, ds,$

which, clearly, is bounded and linear. Then, the null controllability of the control system (3.1) on t means that

$$\text{range } S(t) \subseteq \text{range } V(t). \tag{3.2}$$

Denoting by B(r) the closed ball centered at zero with radius r in a Banach space, by the open mapping theorem, inclusion (3.2) is equivalent to the following one:

$$S(t)B(r(t)) \subseteq V(t)B(1), \tag{3.3}$$

for some constant r(t). This equivalence still holds in the case when V(t) is only a closed and densely defined operator. This allows to consider also boundary control systems.

Define the reachable set at time t, denoted by R(t), as the set of all points x such that there exists a control u in B(1) with y(t,x,u)=0. The reachable set in free time is the union of all R(t) for
positive t and is denoted by R.

Definition The control system (3.1) is said to be admissible null-controllable if all points of X can be transferred to zero in finite time with controls in B(1), i.e., R=X.

In this paper we study estimates of the sets R(t) for t large and in particular we study cases when R=X. As a matter of fact, R is the domain of the minimal time function. Recall that, the minimal time corresponding to x is the infimum of the times taken to transfer x to zero with admissible controls. Of course, here the set of admissible controls is a ball with the center in zero.

Supposing that system (3.1) is null-controllable on any time t in $[0,T_0]$, we provide here remarkable subsets of R(t) for any $t \geq T_0$ and we get estimates of the minimum energy. Moreover, if we know a continuous function r(t) on $[0, T_0]$ satisfying (3.3) for every t, we extend continuously this function on the whole interval $[0, \infty)$ preserving condition (3.3) and having a good growth rate. Further, if the semigroup has a linear growth (in particular, if it is uniformly bounded), then this new function r(t) has the property that tends to infinity when t tends to infinity. Consequently, the control system is admissible null-controllable in this situation. Now we give the main results of this paper.

Results

Theorem

Suppose that there exist T_0 and a function r defined on $[0, T_0]$, with r(0)=0 such that (3.3) holds for any t in $[0, T_0]$. Then, it can be extended on $[T_0 , \infty)$ by

$$r(t)=(r(T_0)/|S(q)|)\sum 1/|S(T_0)|^k +r(q), \quad k=0,1, .. n-1 \tag{4.1}$$

for $t=q+nT_0$ with q in $(0, T_0]$, which satisfies (3.3) for every t in $[T_0, \infty)$.

In particular, if $S(t)$ is a semigroup of contraction, we get

Corollary

Suppose that $|S(t)| \leq 1$ for any $t \geq 0$. Suppose further that there exist $T_0 > 0$ and a continuous and strictly increasing function $r(t)$ defined on $[0, T_0]$, with $r(0)=0$, which satisfies (3.3) on $[0, T_0]$. Then, the extension $r(t)$ defined on $[T_0, \infty)$ by

$r(t)= nr(T_0)+ r(q)$

for $t=nT_0+q$, q in $(0, T_0]$, is continuous and strictly increasing too and satisfies (3.3) on $[T_0, \infty)$. Moreover, $r(t)$ tends to infinity when t tends to infinity.

A similar result can be obtained in case $S(t)$ has linear growth, i.e. $|S(t)| \leq c(t+1)$ for any $t \geq 0$ and some constant $c > 0$. In this case define $r(t)=(1/c)(r(T_0)\sum 1/(kT_0+q+1)+r(q))$, $k=0,1, .. n-1$, for $t=q+nT_0$ with q in $(0, T_0]$.

A finite dimensional example of a continuous semigroup with linear growth, which is not uniformly bounded, is given by $S(t)=[-2t+1 \quad -t;\ 4t \quad 2t+1]$, with t nonnegative, generated by $A=[-2 \quad -1;\ 4 \quad 2]$. It is easy to see that the norm of $S(t)$ is not less or equal to M, for any nonnegative t and any positive M, but satisfies $|S(t)x| \leq c\ (5t+1)\ |x|$ for any x in R^2 and any nonnegative t.

Examples

In this section we give examples of functions $r(t)$ satisfying (3.3).

Example 1 In the finite dimensional setting, when $X = R^n$, in [3, Theorem 2.2], it is provided the function $r(t)$ of the form:

$$r(t)= c \min\{t^n, t\}/\sup\{|S(s)|;\ s \text{ in } [0,t]\},$$

where c is a positive constant and k is an integer such that the well known Kalmann condition is satisfied:

$\text{span}\{BU, ABU, A^{n-1}BU\}=X.$

It is clear that $r(t)$ is strictly increasing with $r(0)=0$. In the case when the semigroup is uniformly bounded, then $r(t)$ tends to infinity when t tends to infinity and, for each x of X, $E(t,x)$ tends to zero for t large.

Example 2 Consider the case of reflexive X, $U=X$ and B is the identity operator.

It is known that the corresponding control system is null-controllable on every $t > 0$ with controls from L^∞ (see, e.g., [14]). More exactly, we have that the ball $B((1/\omega)\ (1-\exp(-\omega t)))$ is a subset of $R(t)$. Therefore, in the case when ω is not zero, we have $r(t)= (1/\omega M)\ (1-\exp(-\omega t))$. If ω is positive then $r(t)$ tends to $1/\omega$ when t tends to infinity. Therefore, the open ball centered in zero and having the radius $1/\omega$ is a subset of R. Take now the case when $\omega = 0$. It is easy to see that $r(t) = t$ is a good function that satisfies (3.3). Moreover, if ω is not strictly positive, then $r(t)$ tends to infinity when t tends to infinity, so we have controllability with vanishing energy and $R=X$.

Example 3 This example refers to the heat equation on a bounded domain (with boundary of class C^2) with homogeneous Dirichlet boundary conditions, in which the control is distributed internally on an open subset.

The considered partial differential equation can be rewritten as an ordinary differential equation in L^2 of the form (3.1). In the L^∞-control, in [5, Proposition 3.2] it is proved that there exists $C>0$ such that (3.3) holds with $r(t)=\exp(-C(1+1/t+t))$, for any positive t. Clearly, r tends to zero when t tends to zero. This function increases on $[0,1]$ and is decreasing otherwise, tending to zero when t tends to infinity. Then, by Theorem 4.1, we provide a better function fot t greater than 1 with the property that it tends to infinity when t tends to infinity. Moreover, we get that E(t,x) tends to zero for t large and for any x in L^2. Consider now the L^2 – control case. In this situation, again in [5, Proposition 3.2] it is proved that there exists C>0 such that (3.3) holds with r(t)=exp(-C(1+1/t)), for any positive t. Clearly, r tends to zero when t tends to zero. This function increases on [0,1] and is decreasing otherwise, tending to exp (-C) when t tends to infinity. Theorem 4.1 provides a better function fot t greater than 1 with the property that it tends to infinity when t tends to infinity.

We recall that the semigroup generated by the Laplace operator subject to Dirichlet boundary conditions in L^2 has M=1 and ω<0.

We mention finally that throughout we supposed that |S(t)| > 0 for each t>0; the case when $S(t_0) = 0$ for some t_0 is not of interest for the problem studied here.

Conclusion

The behavior of the minimum energy for large time does not depend on the control operator. It depends on the state operator under the basic assumption that the system is null controllable at some time. In the case when the control system is null controllable for every t>0, then the minimal time function is continuous in zero. Moreover, the reachable set R is open and the minimal time function is locally uniformly continuous on R. We end this paper pointing out that the results can be extended to systems with boundary controls, both for parabolic and hyperbolic type. In this situation the operator B is not necessarily bounded but there are no differences in reasoning (see [15] for a good reference).

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