PATH INTEGRAL QUANTIZATION OF DISSIPATIVE SYSTEMS

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Abstract

This paper investigated the basic formalism for treating the dissipative Hamiltonian systems within the framework of the canonical method using the path integral quantization. The Hamiltonian treatment of the dissipative systems leads to obtain the equations of motion as a total differential equation in many variables which require the investigation of integrability conditions on the action and the equations of motion. In this formalism, the action integral for dissipative systems is obtained from equations of motion and written in phase space. Besides, the quantization of these systems is investigated using the action to construct the wave function in terms of the canonical phase space coordinates. Two examples are examined, the free particle and the simple harmonic oscillator.

Keywords: Hamilton-Jacobi Equation, Dissipative Systems, Path Integral Approximation

Introduction

The basic idea of the path integral formulation can be traced back by (Norbert Wiener, 1921) who introduced the Wiener integral for solving problems in diffusion and Brownian motion. This idea was extended to the use of the Lagrangian in quantum mechanics by (Dirac, 1933). The complete method was developed by (Feynman, 1948).

The canonical formalism for investigating singular systems was developed by (Rabei and Guler, 1992 Pimentel and Teixeiria, 1996, 1998). In this method, authors obtained a set of Hamilton-Jacobi partial differential equations HJPDEs. In this formalism, the equations of motion are obtained as total differential equations. Depending on this method, the path-integral quantization of constrained Lagrangian systems has been investigated by

(Muslih and Guler, 1997; Rabei, 2000; Muslih, 2002, 2001). Moreover, the quantization of constrained systems has been studied using the WKB approximation (Rabei, 2002; Nawafleh, 2002, 2004; Hasan, 2004). The HJPDEs for these systems have been constructed using the canonical method; the Hamilton-Jacobi functions have then been obtained by solving these equations.

Recently, the quantization of dissipative systems has been studied using the WKB approximation by (Jarab'ah, 2013). The HJPDEs for these systems have been constructed using the canonical method; the Hamilton-Jacobi functions have then been obtained by solving these equations.

This paper is mainly concerned with the Hamiltonian dissipative systems which characterized by dissipative Lagrangians using the canonical approach and then to quantize these systems using the canonical path integral method.

Our present work is organized as follow. In section 2, the canonical path integral formulation of dissipative systems is discussed. In section 3, illustrative examples are examined in detail. The work closes with some concluding remarks in section 4.

2. The canonical path integral formalism of dissipative systems

The Lagrangian formulation for dissipative systems has the following form (Bateman, 1931):

$$L_0 = L(q_i, \dot{q}_i)\, e^{\lambda t}. \qquad i = 1, \ldots, N\,. \tag{2.1}$$

Here $L(q_i, \dot{q}_i)$ represents the first-order Lagrangian of the conservative system.

The term $e^{\lambda t}$ refers to the dissipative system and λ is the damping factor.

The corresponding Euler-Lagrangian equations of motion are obtained from

$$S = \int L_0(q_i, \dot{q}_i)\, dt\,. \tag{2.2}$$

The resulting action S is

$$S(q, \dot{q}, t) = \int e^{\lambda t} L(q, \dot{q})\, dt = \int e^{\lambda t} (p\dot{q} - H_0)\, dt = \int L_0(q, \dot{q}, t)\, dt.$$

using the Hamilton principle:

$$\frac{\partial L_0}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{q}_i}\right) = 0\,. \tag{2.3}$$

Equation (2.3) represents the Euler-Lagrange equation.

The canonical momentum p_i conjugate to the coordinates q_i is given by

$$p_i = \frac{\partial L_0}{\partial \dot{q}_i}. \tag{2.4}$$

Now, we will give a brief review of the Caratheodory's equivalent Lagrangian method (Caratheodory, 1967). The completely equivalent Lagrangian For first-order Lagrangian $L_0(q_i, \dot{q}_i, t)$ can be written as:

$$L_0' = L_0(q_i, \dot{q}_i, t) - \frac{dS(q_i, t)}{dt}, \tag{2.5}$$

The canonical action $S(q_i, t)$ must satisfy

$$\frac{\partial S}{\partial t} = -H_0,$$

(2.6)

The canonical Hamiltonian H_0 is defined as:

$$H_0 = p_i \dot{q}_i - L_0, \tag{2.7}$$

$$p_i = \frac{\partial S}{\partial q_i}. \tag{2.8}$$

The momentum p_0 is given by

$$p_0 = \frac{\partial S}{\partial t}. \tag{2.9}$$

The Hamilton-Jacobi equation can be written as:

$$H_0' = p_0 + H_0 = p_0 + H_0\left(t, q_i; p_i = \frac{\partial S}{\partial q_i}\right) = 0. \tag{2.10}$$

In the canonical method the action function and equations of motion are written as total differential equations as follows:

$$dq_i = \frac{\partial H_0'}{\partial p_i} dt, \tag{2.11a}$$

$$dp_i = -\frac{\partial H_0'}{\partial q_i} dt, \tag{2.11b}$$

$$dZ = \frac{\partial S}{\partial t} dt + \frac{\partial S}{\partial q_i} dq_i = (-H_0 + p_i \frac{\partial H_0'}{\partial p_i}) dt. \tag{2.11c}$$

Where $Z = S(q_i, t)$ is the canonical action (the Hamilton-Jacobi function).

Now, the canonical action integral is obtained in terms of the canonical coordinates. H_0' can be interpreted as the infinitesimal generators of canonical transformations given by parameters t and q_i. In this case, the path integral representation may be written as

$$K(q,\dot{q}) = \int \left[\exp i \left\{ \int (-H_0 + p_i \frac{\partial H_0'}{\partial p}) dt \right\} \right] dp_i dq_i . \quad (2.12)$$

3. Examples

3.1 Free particle

As a first example, let us consider one-dimensional Lagrangian in the presence of dissipation.

$$L_0(q,\dot{q},t) = \left(\frac{1}{2} m\dot{q}^2 \right) e^{\lambda t} . \quad (3.1)$$

The canonical momentum is

$$p = \frac{\partial L_0}{\partial \dot{q}} = m\dot{q}\; e^{\lambda t} . \quad (3.2)$$

This equation can readily be solved to give

$$\dot{q} = \frac{p}{m} e^{-\lambda t} . \quad (3.3)$$

The canonical Hamiltonian is given by

$$H_0 = p\dot{q} - L_0 . \quad (3.4)$$

Now, the usual Hamiltonian is written as:

$$H_0 = \frac{p^2}{2m} e^{-\lambda t} \quad (3.5)$$

The reduced form of the Hamilton-Jacobi equation is

$$H_0' = p_0 + \frac{p^2}{2m} e^{-\lambda t} . \quad (3.6)$$

By using Eqs. (2.11a, b), we can obtain

$$dq = \frac{p}{m} e^{-\lambda t} dt, \quad (3.7)$$

$$dp = 0 \quad (3.8)$$

Making use of Eqs. (2.11c), one can find the action integral in phase space as:

$$dZ = \frac{p^2}{2m} e^{-\lambda t} dt. \quad (3.9)$$

Now using Eq. (3.3), one can write the original action in terms of $\dot{q}$ is:

$$Z = \int \frac{\dot{q}^2 m}{2} e^{\lambda t} \, dt. \tag{3.10}$$

Using Eq. (2.12), the path integral representation can be written as follow:

$$K(q,\dot{q}) = \int \left[\exp i \left\{ \int \frac{m\dot{q}^2}{2} e^{\lambda t} \right\} dt \right] dp_i dq_i. \tag{3.11}$$

3.2 Damped Harmonic Oscillator

The following Lagrangian is considered for a damped harmonic oscillator (Bateman, 1931):

$$L_0(q,\dot{q},t) = \left(\frac{1}{2} m\dot{q}^2 - \frac{1}{2} m\omega^2 q^2 \right) e^{\lambda t}, \tag{3.12}$$

The linear momentum is given by

$$p = \frac{\partial L_0}{\partial \dot{q}} = m\dot{q} \, e^{\lambda t}. \tag{3.13}$$

The canonical Hamiltonian is given by

$$H_0 = p\dot{q} - L_0. \tag{3.14}$$

Now, the Hamiltonian is written as:

$$H_0 = \frac{p^2}{2m} e^{-\lambda t} + \frac{1}{2} m\omega^2 q^2 \, e^{\lambda t}. \tag{3.15}$$

One can write the HJE in terms of phase space

$$H_0' = p_0 + \frac{p^2}{2m} e^{-\lambda t} + \frac{1}{2} m\omega^2 q^2 \, e^{\lambda t}. \tag{3.16}$$

By using Eqs. (2.11a, b), we can obtain

$$dq = \frac{p}{m} e^{-\lambda t} dt \,, \tag{3.17}$$

$$dp = -m\omega^2 q \, e^{\lambda t} dt \tag{3.18}$$

Making use of Eqs. (2.11c), the action integral in phase space can now be obtained as:

$$dZ = \left[\frac{p^2}{2m} e^{-\lambda t} - \frac{1}{2} m\omega^2 q^2 \, e^{\lambda t} \right] dt. \tag{3.19}$$

Using Eq. (3.13), one can write the original action in terms of q_i and $\dot{q}_i$.

$$Z = \int \left(\frac{m\dot{q}^2 - m\omega^2 q^2}{2}\right) e^{\lambda t}\ dt. \qquad (3.20)$$

Finally, the path integral representation can be written as:

$$K(q,\dot{q}) = \int \left[\exp i \left\{ \int \left(\frac{m\dot{q}^2}{2} - \frac{1}{2} m\omega q^2\right) e^{\lambda t} dt \right\} \right] dp_i dq_i. \qquad (3.21)$$

4. Conclusion

This work has aimed at shedding further light on the dissipative systems within the framework of the canonical method for treating dissipative dynamical systems. The aim is twofold: first, to formulate the theoretical framework involved; and, secondly, to shed some light on its applications and illustrate this formalism with some examples. The non-natural Lagrangian are reduced to natural Lagrangian by using separation of variables and this procedure leads to regular systems which can be treated by the canonical method, and quantized by the canonical path integral. The action function that related to the extended Lagrangian has been obtained. The formulation for the path integral quantization of dissipative systems is investigated. The action function and equations of motion are written as total differential equation in many variables. Two illustrative examples have been studied in detail.

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