SPIRALITY OF VANISHING HELICITY FLOWS IN SOLID TORI DOMAINS

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Abstract
　　Ideal nature of inviscid flows and persistency of field lines found a nice bridge between their dynamical properties and topological considerations which on the top of them is knottedness.
Helicity as the simplest invariance in topological dynamics, lies in the cross road of these disciplines, both conceptually and practically. Spirality is a Lagrangian invariant that targeted in this paper which imparts the behaviors of each field line. In the present article a necessary and separately a sufficient condition are obtained for vanishing the spirality via a certain gauge transformation in a zero helicity flow.
Geometrical interpretations of these conditions are presented and the possible relation between the existence of such a gauge and occurring singularities in vortex dynamics are discussed.

Keywords: Helicity, spirality, topological flows

Introduction:
　　The influence of topology in physics for the first time emerged in Gauss's studies (in 1833) to calculating the work performed on a magnetic pole moving trough a closed path in the present of a loop of electric current which leads to defining the linking number that regarded as a measure of entanglements of two closed curves (Nash, 1999). Gauss realize that the linking number of two separate curves K_1 and K_2

$$Link\left(K_1, K_2\right) = \frac{1}{4\pi} \int_{K_1 \times K_2} \frac{dx}{ds} \times \frac{dy}{dt} \cdot \frac{x-y}{|x-y|^3} ds dt$$

is constant under continuous deformations of curves and it became the first tool to distinguish two non-equivalent links (disjoint union of closed curves) (Gauss, 1833).
　　This can also applies to closed curves (knots) by computing the linking number of a closed curve with itself that named writhing numberor

self-linking   number,   which   was   discussed   by   Călugăreanu $(Wr\left(K\right) = Link\left(K, K\right)$(Călugăreanu 1959,1961)).

The second trace of topological ideas in science appeared in 1857 when Helmholtz analyzed the time evolution of vortex lines and tubes driven by an inviscid fluid. He discovered that by the evolution the vortex lines, tubes and rings, up to deforming the shape, remain vortex lines, tubes and rings and their strengths also remain constant. This property that described by Helmholtz as the frozzness of vortex lines into the fluid, later interpreted by Poincaré as, the dynamical properties of the vorticity field, up to a change of variables, remains invariant under the time evolution (Helmholtz, 1858), (Poincaré, 1893).

After a decade, the third echo was taken in Kelvin's works (in 1869) which introduced a model using knots and links to describe atoms and chemical elements. While most of the fundamental physicists were divided to: Supporters of corpuscular theory and adherents of wave's model for matter,

Kelvin originated a third way from vortex dynamics to merge those two dominant models. Kelvin and Tait tried to tabulating all possible knots to making a table of elements and to find the reason of quantized wavelengths in atoms absorbtion and emission. In 1885 Tait made a chart of knots with up to ten crossings, and conjectured that the fewest possible crossings of reduced diagram of an alternating knot is an invariant (Atiyah, 1990).

In this model diversity of elements was translated to variety of knots and degree of excitation attributed to the frequency of vibration of the vortex atom, and chemical bonds of atoms in a molecule taught to mirror the linkage of knots in a link. The topological invariance of knots and links under the deformations were exposed the stability of matter. Another inspiring for Tait was directly from Helmholtz results in vortex motion that leads to idea of his smoke machine (Gambaudo, 2006). Kelvin was conscious of resemblance between fluid dynamics and electromagnetism and Maxwell was interested to interpolate these ideas and specially Gauss linking number in electricity and magnetism. He presented three allowed methods to move the crossing of the curves for handle a knot to another equivalent knot that called Reidimeister moves in 1910. Indeed, Reidimeister showed that these operations are sufficient to manipulate knots and links to their mimics. However, some experimental data did not support the vortex atoms model and this failure lost physicist's interest in knot theory and topological aspect of dynamics for a while. In 1920's attention to the Tait Conjectures and the generalproblems of classification of knots, develops in knot invariants by

Alexander and advances of braid theory by Artin revived knot theory in purelymathematical feature.

One century after constituting the vortex dynamic, Woltjer published some papers (in 1935) on time evolution of the magnetic fields. He was interested in cosmic magnetic fields in Crab Nebula. In ideal MHD a governing equation obtained by the combination of Faraday's equationand the ideal conductivity condition as:

$$\frac{\partial B}{\partial t} = \nabla \times \left(u \times B \right),$$

The magnetic field B is divergence-free, and let B assumed in a domain Ω in 3-space and tangent to the boundary $\partial\Omega$ and let A be the vector potential of B satisfies $B = \nabla \times A$. Woltjer showed the following integral that later called helicity is a constant of motion

$$\int_{\Omega} A \cdot B d^{3}x\,,$$

when the domain Ω moves by plasma. The proof comes from the fact, the magnetic field is frozen into the plasma as can be seen from the governing equations (Chandrasekhar, 1958), (Woltjer, 1958). Mathematically, one can say the magnetic field push-forwarded by the transform map that changes Ω by the time evolution.

Historically, Elsasser (1956) was a first who recognized that the integrand (helicity density) in ideal MHD, in a particular gauge for A, is a frozen-in quantity.

One of the significant results of helicity provided by using Poisson integral formula and Poincaré inequality is that there is a positive constant R such that:

$$\left| \int_{\Omega} A \cdot B d^{3}x \right| \leq R \int_{\Omega} B \cdot B d^{3}x\,.$$

The above formula bounded the helicity by the energy of magnetic field and also provided a lower bound for the energy of magnetic field.

In 1961 Moreau introduced a fluid dynamical analogous for helicity as

$$H = \int_{\Omega} u \cdot \omega d^{3}x\,, \qquad (1)$$

where u is the velocity and $\omega = \nabla \times u$ is the vorticity of the fluid and Ω is a domain in 3-space (Moffatt, 1992,2001). It is clear the vorticity is divergence-free and it is assumed to be tangent to the boundary $\partial\Omega$. In an isentropic ideal flow the equation of motion (Euler equation) is written as

$$\frac{Du}{Dt} = -\nabla w\,, \qquad (2)$$

where w is the thermodynamical enthalpy per unit mass. This equation will describe the system completely by adding the continuity equation which

specifies conservation of mass. By taking the curl of above equation and employing the identity $\nabla \times (A \times B) = (B \cdot \nabla)A \text{-} (A \cdot \nabla)B + (\nabla \cdot B)A \text{-} (\nabla \cdot A)B$ we arrive at

$$\frac{\partial \omega}{\partial t} = \nabla \times (u \times \omega), \qquad (3)$$

This equation is exactly similar to what that explained for magnetic field in MHD. Therefore, using the continuity equation with some calculations shows that the vorticity field is driven by the velocity field

$$\frac{D}{Dt}\left(\frac{\omega}{\rho}\right) = \left(\frac{\omega}{\rho} \cdot \nabla\right)u,$$

Frozenness of the vorticity (divided by density) into the fluid, geometrically interpreted to its pushforwardness by time evolution $\omega_t / \rho_t = (h_t)_* (\omega_0 / \rho_0)$ where h_t is the map from the initial domain to the domain at t .

In general case of divergence-free vector fields in a domain $\Omega \subset \mathbf{R}^3$ the helicity is preserved under the volume-preserving diffeomorphisms on Ω . A famous example is when the vector field be confined to two narrow linked flux tubes. One may taught the tube as a thin tubular neighborhood of a closed curve K_i and the vector field inside, has the flux Φ_i . Assumed also there is not any linkage between the trajectories in each tube. In this case the helicity of the vector field is given by

$$H = 2\text{Link}(K_1, K_2)\Phi_1\Phi_2.$$

The topological interpretation of helicity in this situation revealed by Moffatt (in 1969) that mentioned this integral measures the average mutual linkage of the field lines, and so the complexity of the field. Here, for minimizing the energy of the vector field, trajectories have to be shorter, that leads to a fattening of the solid tori when the preserved measure by the diffeomorphisms is the volume (Moffatt, 1969), (Contreras, 1999).

V. I. Arnold in 1973 proposed an invariant, later called helicity, for null-homologous fields on 3-manifolds that equipped with volume element and showed it is preserved under the action of volume-preserving diffeomorphisms. He made an ergodic expression of this helicity for divergence-free vector fields on simply connected domains with not necessarily closed or confined with tori trajectories, that is, the average self-linking number of trajectories of a field is coincide with the helicity of the field.

Arnold also discussed on upper bounding of helicity by the energy and showed the constant in the inequality can be the maximum eigenvalue of the inverse of the curl operator (Arnold, 1986, 1998), (Khesin, 2005).

The recent concept inspired Cantarella et al. to define the Biot-Savart helicity of a divergence-free vector field V on a compact domain $\Omega \subset \mathbf{R}^3$ by the following integral

$$H\left(V\right) = \frac{1}{4\pi}\int_{\Omega\times\Omega} V\left(x\right)\times V\left(y\right).\frac{x-y}{\left|x-y\right|^3}dx^3 dy^3,$$

that can be envisaged as a extension of gauss linking number in a glance. So, it became a standard average measure of how much the field lines are knotted and linked. In helicity bounding inequality with this helicity, R is the radius of a round ball having the same volume as Ω (Cantarella, 2000,2001) .

The difference between the Biot-Savart helicity and the helicity in physics for the vector fields in non-simply connected domains was investigated in (Sahihi, 2014) Interested reader finds the helicity in relativistic fluid in (Eshraghi, 2003) the modern versions of helicity in quantum fluids in (Bambah, 2006), (Mahajan, 2006) and the applications of helicity in optics can find in (Kedia, 2010).

In this paper we first review the conversation of helicity in Euler description of isentropic ideal fluid, and then repeat this task in Lagrangian notion that leads to defining spirality. It will be find that the helicity is conserved under a certain gauge transformation.

The main part of this work is in deal with a Lagrangian invariant called spirality and its correspondence to topology and geometry of the fluid flows.

Then, by concentrating on vanishing helicity flows in non-simply connected domains of fluid some of topological properties of flows that reflected in spirality and its probable relation with finite time singularity will be discussed. The simpler case of this inquiry has been done before in. Doing this job needs employing the Hodge decomposition Theorem for splitting the vector fields to desire terms.

Spirality and gauge tranmformations

Suppose a compact domain in 3D Euclidian space with smooth boundary, filled by isentropic ideal fluid, in which the domain and its boundary are allowed to be disconnected. let a fluid particle initially (at $t=0$) located at the point x of fluid filled domain Ω_0. The time evolving of the fluid in the domain can be presented by such a map $h_t : \Omega_0 \to \Omega_t$, so at time t, the journey of this particle ends to one point like y_t in Ω_t. It is clear that the position of the particle $y_t = h_t\left(x\right)$ is a function of time and initial location, and the velocity of fluid in Eq. (2) is the derivation of y_t.

$$u_t = \dot{y}_t = \frac{\partial}{\partial t} h_t(x)\Big|_{x=(h_t^{-1})(y_t)},$$

where the dot means

$$\frac{D}{Dt} := \left(\frac{\partial}{\partial t}\right)_{y_t} + u_t \cdot \nabla_{y_t}.$$

Here, the Jacobian of h_t shows by J_t and one can write $d^3 y_t = J_t d^3 x$, so the preserved mass element is

$$dm = \rho_t d^3 y_t = \rho_0 d^3 x \qquad (4)$$

J_t also is equal to the ratio of initial and final densities ρ_0/ρ_t. Conservation of helicity arises from Eq. (2) and Eq. (4). This result can be rewritten as

$$H = \int_{\Omega_t} u_t \cdot \omega_t d^3 y_t = \int_{\Omega_0} u_0 \cdot \omega_0 d^3 x = const.$$

Inversely, one can obtain the helicity conservation in Lagrange description. By multiplying the i th component of Euler equations of motion Eq. (2) by y_{ti}/x_j one can define

$$v_j = u_i \frac{y_{ti}}{x_j},$$

where the summation over i is dummy.

$$\left(\frac{\partial v}{\partial t}\right)_x = \frac{dv}{dt} = -\nabla_x \left(w - \frac{u^2}{2}\right).$$

An ansatz for the above equation is as follows

$$v(x, t) = v_0(x) + \nabla_x \varphi, \qquad (5)$$

where φ is the Bernoulli's function that satisfied

$$\frac{d\varphi}{dt} = \frac{u^2}{2} - w. \qquad (6)$$

The arbitrary function $f(t)$ depends only on time and not on position in fluid. Suppose $f(t)$ and $\varphi|_{t=0}$ to be zero, inverse of Eq. (5) obtaind as

$$u = u_{0i}(x)\nabla x_i + \nabla_x \varphi,$$

where $x = x(y_t, t)$ and $\nabla = \nabla_{y_t}$. Now let

$$\gamma = u - \nabla\varphi,$$

and call it impulse density function in respect to (Russo, 1999). The curl of this vector is clearly the vorticity of fluid. Dividing γ and ω yields

$$\gamma \cdot \omega = \gamma \cdot \nabla \times \gamma = v_{0i}\nabla x_i \cdot \left(\nabla v_{oj} \times \nabla x_j\right) = \varepsilon_{ijk}\det\left(\frac{\partial x_m}{\partial y_{tn}}\right)v_{oi}\left(\frac{\partial v_{oj}}{\partial x_k}\right).$$

The determinant is the inverse of J_t or the ratio of final and initial densities. Now a lagrangian invariant can defines as

$$Sp\left(x\right) = \frac{\gamma \cdot \omega}{\rho}, \qquad (7)$$

which depends only on initial position, so remains constant for a fluid particle long the trajectory and obviously its invariance leads to the conservation of helicity. On the certain conditions that assumed in this paper, the frozzness of vorticity guaranties that ω_t which was divergence-free and tangent to the boundary at $t = 0$ holds these properties forever. Therefore

$$H = \int_\Omega u \cdot \omega d^3 y = \int_\Omega \gamma \cdot \omega d^3 y = const.$$

Moreover, a gauge transformation as

$$\varphi \longrightarrow \tilde{\varphi} = \varphi + \varphi_0(\mathrm{x}),$$

where $\varphi_0\left(x\right)$ is a time independent function of initial position brings $Sp\left(x\right)$ to a new value $\tilde{Sp}\left(x\right)$ but holds the helicity. In some references especially in incompressible fluid, $\gamma \cdot \omega$ called spirality.Indeed in incompressible case $\gamma \cdot \omega$ is lagrangian invariant and plays the rule of spirality.

From now, we concentrated on the flows with zero helicity and try to find a topological interpretation of a question: In what kind of ideal flows an appropriate gauge transformation takes the spirality to zero. The important point in this question may be in deal with finite time singularity problem. In this case the vorticity vector field that was initially smooth, in finite set of points and times bounces to infinity. When the spirality can not be transformed to zero by any gauge, γ and ω can not be orthogonal everywhere and so the contingency of singularity seems to be reduced. When there is a gauge for vanishing the spirality γ and ω has more freedom to be perpendicular. There is not any proof for this conjecture and neither both necessary and nor sufficient condition for satidfiyng this condition. For flows in simply connected region a necessary condition and a sufficient condition for the existance of this gauge presented in (Eshraghi, 2005). In the general case of non-simply connected domains, the same conditions hold. The proof of the necessary condition does not posses any additional difficulty and the sufficient condition needs to a new proof with more details. The vital tool in the second proof is the Hodge decomposition theorem, which is the general

version of the Helmholtz decomposition theorem for the case of non-simply connected domains.

The more general version of this theorem for differential forms on manifolds can be find in (Warner, 1983). In the case of real vector fields insubspace of a 3-space it reduces to following the expression.

Hodge decomposition Theorem. Suppose Ω be a compact domain in $\mathbf{R}^3$ with piecewise smooth boundary and $\Gamma(\Omega)$ be the infinite dimensional space of smooth vector fields on Ω, then, $\Gamma(\Omega)$ uniquely splits into the direct sum of mutually orthogonal (in the L^2 sense) subspaces as:

$$\Gamma(\Omega) = FK(\Omega) \oplus HK(\Omega) \oplus CG(\Omega) \oplus HG(\Omega) \oplus GG(\Omega), \qquad (8)$$

where

$$FK(\Omega) = Fluxless\ knots = \left\{ V \in \Gamma(\Omega) \middle| \nabla \cdot V = 0, V \cdot n = 0, \ all\ \text{interior}\ fluxes = 0 \right\},$$

$$HK(\Omega) = Harmonic\ knots = \left\{ V \in \Gamma(\Omega) \middle| \nabla \cdot V = 0, \nabla \times V = 0, V \cdot n = 0 \right\},$$

$$CG(\Omega) = Curly\ gradients = \left\{ V \in \Gamma(\Omega) \middle| V = \nabla \phi, \nabla \cdot V = 0, \ all\ boundary\ fluxes = 0 \right\},$$

$$HG = Harmonic\ gradients = \left\{ V \in \Gamma(\Omega) \middle| V = \nabla \phi, \nabla \cdot V = 0, \ \phi\ locally\ cons.\ on\ \partial\Omega \right\},$$

$$GG(\Omega) = Grounded\ gradients = \left\{ V \in \Gamma(\Omega) \middle| V = \nabla \phi, \ \phi \middle|_{\partial\Omega} = 0 \right\},$$

The harmonic subspace, includes $HK(\Omega)$ and $HG(\Omega)$, have definitely finite dimensions and others are not generally finite dimensional.This finiteness relates to the homology groups of the bachground domain, that showed in following isomorphisms

$$HK(\Omega) \cong H_1(\Omega, R) \cong H_2(\Omega, \partial\Omega : R) \cong R^{total\ genus\ of\ all\ components\ of\ \partial\Omega},$$

$$HG(\Omega) \cong H_2(\Omega, R) \cong H_1(\Omega, \partial\Omega : R) \cong R^{\#(\partial\Omega) - \#(\Omega)}.$$

where # refers to the number of components. The second isomorphisms in the above relations obtain from the Poincaré duality. The last isomorphisms for a compact 3-domain Ω was proved in (Cantarella, 2002) and also there is an alternative proof for these isomorphisms in (Sahihi, 2014).

Spirality and Geometry of the flow

Consider a smooth boundary domain Ω in fluid and assume the vorticity field Ω is tangent to the boundary. Let the domain contains only some vorticity closed field lines such that the vorticity does not vanishes on them. Suppose C be such a closed curve ($\omega|_C \neq 0$) and dl be the normal

tangent vector along C $(dl = \dfrac{\omega}{|\omega|})$ and recalling the notation in the case a gauge transformation to vanishing the spirality is exist

$$\gamma \to \tilde{\gamma} = \gamma + \nabla_x \phi,$$

$$Sp\left(x\right) = \frac{\gamma \cdot \omega}{\rho} \to \tilde{S}p\left(x\right) = \frac{(\gamma - \nabla_x \phi) \cdot \omega}{\rho} \equiv 0.$$

Since spirality is a lagrangian invariant, the argument given for $t = 0$ does also work for any time t. Hence, on domain Ω_0, dividing the righthand side of the second transformation above by the norm of vorticity and taking the integral along the curve C_0 leads to

$$\oiint_{C_0} u_0 \cdot dl = \oiint_{C_0} \nabla_x \varphi_0 \cdot dl = 0.$$

This integral shows the circulation around the curve C_0 that by using Stokes' theorem it becomes

$$\int_{\Sigma_0} \omega_0 \cdot dS = 0,$$

where Σ_0 is a Seifert surface (though oriented) bounded by closed curve $C_0 = \partial \Sigma_0$, if entirely lies in the domain Ω_0. The last equation imposed, a *necessary condition* states that the flux passing through any embedded (in Ω_0) Seifert surface of a nonsingular closed vortex line should be zero. For example, a system contains just four vortex lines that appears in two separate Hopf link with different orientations, although has a zero helicity but does not satisfy this necessary condition and so can not be converted to zero spirality be any gauge transformation. To extract a sufficient condition, the Hode decomposition should examines to splitting the impulse density field as

$$\gamma_0 = \gamma_0^{FK} + \Upsilon_0 + \gamma_0^{CG} + \gamma_0^{HK} + \gamma_0^{GG}, \qquad (9)$$

where Υ_0 stands for the harmonic knot part of the field and in the other terms each superscript cites the relevant subspace. The last three terms belong to the gradient part of $\Gamma(\Omega_0)$ and so for briefness one can write $\nabla_x \varphi_o^G = \gamma_0^{CG} + \gamma_0^{HK} + \gamma_0^{GG}$ and due to the mutual orthogonality and uniqueness of functional components of $\Gamma(\Omega_0)$ we have $\gamma_0^{FK} = u_0^{FK}$ and $\Upsilon_0 = u_0^{HK}$. Therefore Eq. (9) finally converts to

$$\gamma_0 = u_0^{FK} + u_0^{HK} + \nabla_x \varphi^G,$$

and the vorticity becomes $\omega_0 = \nabla_x \times \omega_0^{FK}$. The gauge transformation can change the gradient part of γ_0 without affecting other terms. Hence to cancel the spirality an appropriate choice of ϕ_0 in $\gamma_0 - \nabla_x \phi_0$ is

$$\phi_0 = \varphi_0^G,$$

that always holds $\nabla_x \left(\phi_0 - \varphi_0^G\right) \cdot \omega_0 = 0$. Therefore, the spirality transforms to zero only if

$$\left(u_0^{FK} + u_0^{HK}\right) \cdot \omega_0 = \left(u_0^{FK} + u_0^{HK}\right) \cdot \left(\nabla_x \times u_0^{FK}\right) = 0.$$

So one can define $u_0^K = u_0^{FK} + u_0^{HK}$. and write the sufficient condition as

$$u_0^K \cdot \left(\nabla_x \times u_0^K\right) = 0.$$

In the above equation the conditions of Frobenius theorem (generally Darboux theorem) are provided and so it leads to integrability of the field and locally existing a smooth surface Σ such that at each point

$$u_0^K \parallel \hat{n}_\Sigma,$$

meaning that Σ is normal to the field u_0^K.

Conclusion:
Helicity is known as a gadget to reveal some of the dynamical details and topological subtleties of physical vector fields. As we get closer, a less known lagrangian invariant (spirality) can be engaged to mine more facts about the entwined field lines and relevant physical properties. In the present work we generalized what was obtained in (Esh, 2005), the necessary condition and the sufficient condition to find a gauge transformation that makes the spirality zero in the case of vanishing helicity flows on non-simply connected domains. It turns out that the sufficient condition is the hypotheses to integrability (in the sense of Frobenius). The former was almost same as what presented in (Eshraghi, 2005) while the latter had a new argument and a new result. It was mentioned how this gauge problem can deal with occurring singularities and especially finite time singularities.

Finding a proof for this conjecture seems to be far and there is still no necessary and sufficient condition that guarantees the existence of this gauge transformation. For the case of non vanishing helicity there is still no considerable information, but since the spirality has the role of contact form on 3D domains, it seems contact topology and generally confoliations theory may probes more details in future investigations.

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