ON THERMAL EXPLOSION ARISING FROM TIME-DEPENDENT GRAVITATIONAL DIFFERENTIATION AND RADIOACTIVE DECAY IN THE EARTH'S INTERIOR

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Abstract

Problem statement: The combustion processes in the interior of the earth during the gravitational differentiation and radioactive decay processes are characterized by ignition and explosion. The two heat sources are in this paper, considered as time-dependent.

Approach: The study investigates the existence of unique solution of the new model and examines the effect of activation energies ratio on the reaction. The criteria for a similarity solution of the resulting equations are established. The resulting ordinary differential equations are solved numerically by shooting method.

Results and Conclusion: The results show that with time-dependent gravitational differentiation and radioactive decay processes, the problem has a unique solution and the activation energies ratio has an implication on the reactions in terms of heat release.

Keywords: Activation energy, Gravitational Differentiation , Temperature, Radioactive Decay

Introduction

Researchers have shown that there are two major sources of heat that constitute the exothermic regimes in the interior of the earth; Gravitational differentiation and Decay of radioactive elements. The thermal explosions do occur in the interior of the earth as a result of both processes.

In the early stages of planetary build up, the earth was much less compact than what it is today. This build-up process led to more gravitational attractions which force the earth to contract into smaller volume. During the gravitational differentiation, the potential energy generated becomes heat energy due to viscous dissipation. Also, radioactive elements are inherently unstable; The unstable Uranium isotope (Uranium-238) slowly decay to Lead - 206 and the radioactive decay processes continue. They break down over time to more stable forms and release intense heat as by-product of the chain reaction. This heat is continually radiated outward through several concentric shells that form the solid portion of the planet.

Zel'dovich et. al (1980) and Vityazev (2004) explained that thermal processes that occur in the earth interior differ from characteristic thermal explosion but they are analogous. Popoola et.al (2005) and Popoola (2007) also established that although multiple steps are involved in reaction but two major steps are basically involved, and these include chemical decomposition and combustion process. Explosion for instance, generally results from two exothermic reactions; one step follows the other in very rapid succession depending on the activation energies of the reactions.

Popoola (2011) investigated the effect of activation energies on thermal explosion that occurs in the interior of the earth during gravitational differentiation and decay of radioactive substances. The unsteady, steady and homogeneous reactions were modelled and investigated. Results on blow -up (Buckmaster and Ludford, 1992) were also obtained. Theorems on the existence of unique solution were formulated and proved. The criteria for a blow up to occur in the chain reaction, were established. The analytical and numerical results showed that activation energies have different implications in terms of heat release.

Ayeni et.al (2008) revisited the theory of evolution of the earth. The effect of gravitational differentiation in the separation of heavier material forming the earth's core from Silicates in the extended and heated area was studied. Previous literatures focused on small and large thermal conductivities but we focused on all orders of thermal conductivity. The numerical solution of the energy equation was provided by shooting method. The previous results in the literatures were special cases of the new results in that paper.

Ayeni et. al(2007a) also investigated the effect of radioactive heat source and gravitational differentiation on thermal explosion in the evolution of the earth. The resulting energy equation associated with the earth evolution was solved by shooting method. The authors showed that the critical temperature which signifies the onset of thermal instability due to gravitational differentiation depends linearly on the intensity of radioactive heat source.

Ayeni et.al. (2007b) presented some remarks on thermal explosion in the early evolution of the earth. The steady state energy equation associated with the earth evolution was solved by shooting method. The paper established the criteria for the steady solution of energy equation associated with the early evolution of the Earth. We were able to show that even when the thermal conductivity due to gravitational differentiation and ordinary thermal conductivity are comparable, a steady thermal solution exists under specified conditions.

Ayeni et. al.(2006) presented some remarks on thermal explosion in the early evolution of the earth. The paper considered the unsteady state energy equation associated with the earth evolution, non-dimensionalize the same equation and obtain a new equation for highly exothermic reaction. The criteria for the occurrence of thermal runaway were established.

In this paper, the two major sources of heat that constitute the exothermic regimes are considered as time dependent. It should be noted that previous studies considered them as fixed processes. The model formulated by Popoola (2011) is revisited under the assumption that gravitational differentiation and radioactive decay processes are time-dependent. The effect of activation energy ratio on the resulting equation is investigated.

Mathematical Formulation

Following Popoola (2011) the dimensionless thermal conductivity equation governing the generation of heat by GD and the decay of radioactive elements is given by

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial \xi}\left(1 + P_{eo} e^{\frac{(1+\alpha)\theta}{n}}\right)\frac{\partial \theta}{\partial \xi} + \Gamma_d \, e^{(1+\alpha)\theta} + \Gamma_r \qquad (1)$$

together with the initial and boundary conditions

$$\theta(\xi,0) = 0, \qquad \theta(-1,\tau) = \theta(1,\tau) = 0 \qquad (2)$$

Remarks:

(i). When $\alpha = 0$, the unsteady equation formulated by Vityazev (2004) is obtained.

(ii). Since the processes of gravitational differentiation and decay of radioactive elements are time dependent, we take

$$\Gamma_r = \frac{\varepsilon_r h^2 E_1}{4\lambda R T_o^{\,2}} = \Gamma_r(t), \quad \Gamma_d = \frac{\Delta \rho g c v_o h^2 E_1}{4\lambda R T_o^{\,2}} = \Gamma_d(t) \qquad (3)$$

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial \xi}\left(1 + P_{eo} e^{\frac{(1+\alpha)\theta}{n}}\right)\frac{\partial \theta}{\partial \xi} + \Gamma_d(\tau) e^{(1+\alpha)\theta} + \Gamma_r(\tau) \qquad (4)$$

where the following parameters are defined as;

θ	Non-dimensional temperature
τ	Non-dimensional time variable
ξ	Non dimensional space variable.
P_e	Péclet number
α	Ratio of activation energies

E_1	Activation energy of the first step reaction
E_2	Activation energy of the second step reaction
ρ	Density of the reactant
T_o	Initial temperature of the reactant
ε_d	Rate of heat generation by Gravitational Differentiation
ε_r	Rate of heat generation by radioactive source
Γ_d	Non dimensional term for Gravitational Differentiation
Γ_r	Non dimensional term for radioactive source
R	Universal gas constant
v_o	Velocity of inclusions
λ	Ordinary thermal conductivity

Similarity Transformation

Let $\theta(\xi,\tau)=\varphi(\eta)$ *such that* $\eta=\frac{\xi}{\tau^{\gamma}}.$ (Ayeni et.al, 2006) (5)

Case 1: $P_{eo}e^{\frac{(1+\alpha)\theta}{n}} >> 1,.$

We obtain the equation

$$\frac{d}{d\eta}\left(1+\ p_e e^{\frac{(1+\alpha)\varphi}{n}}\right)\frac{d\varphi}{d\eta}-\frac{\eta}{2}\frac{d\varphi}{d\eta}+\ \Gamma_d e^{(1+\alpha)\varphi}\ +\ \Gamma_r=\ 0$$

$$\Rightarrow\left(1+P_{eo}e^{\frac{(1+\alpha)\varphi}{n}}\right)\frac{d^2\varphi}{d\eta^2}+\left(\frac{(1+\alpha)P_{eo}}{n}e^{\frac{(1+\alpha)\varphi}{n}}\right)\left(\frac{d\varphi}{d\eta}\right)^2-\ \frac{\eta}{2}\frac{d\varphi}{d\eta}+\Gamma_d\ e^{(1+\alpha)\varphi}+\ \Gamma_r=0 \tag{6}$$

Satisfying $\varphi(-1)=\varphi(1)=0$ (7)

Remark: For similarity, $\gamma=\frac{1}{2},\ \ \Gamma_r(t)=\frac{\Gamma_r}{t},\ \ \ \Gamma_d(t)=\frac{\Gamma_d}{t}$ (8)

Existence of Unique Solution:

Theorem 1: *For* $0\le\alpha\le N$, $-1\le y_1\le 1$, $N,\Gamma_r,\Gamma_d,n,\mathrm{P}_{eo}>0$, *problem (6) which satisfies conditions (7) and for which* $\varphi'(-1)$ *is fixed, has a unique solution.*

$$\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}=\begin{pmatrix}\eta\\\varphi\\\varphi'\end{pmatrix} \tag{9}$$

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 1 \\ x_3 \\ -\dfrac{\left[\dfrac{(1+\alpha)p_{e_0}}{n}\exp\left(\dfrac{(1+\alpha)y_2}{n}\right)(y_3)^2 - \dfrac{y_1 y_3}{2} + \Gamma_d \exp((1+\alpha)\ y_2) + \Gamma_r\right]}{\left(1 + p_{e_0}\exp\left(\dfrac{(1+\alpha)y_2}{n}\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \psi_1\ (y_1,\ y_2,\ y_3) \\ \psi_2\ (y_1,\ y_2,\ y_3) \\ \psi_3\ (y_1,\ y_2,\ y_3) \end{pmatrix} \qquad (10)$$

satisfying the initial conditions

$$\begin{pmatrix} y_1(-1) \\ y_2(-1) \\ y_3(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -\lambda_g \end{pmatrix} \qquad \text{s} \qquad (11)$$

Remark: λ_g is guessed such that the boundary condition $y_2(1) = 0$.

Theorem 2:

For $0 \le \alpha \le N$, $-1 \le y_1 \le 1$, $0 \le y_2 \le M$, $\lambda_g \le y_3 \le -\lambda_g^*$, $M, N,\ \Gamma_r, \Gamma_d, n,\ P_{eo} > 0$, *the functions* $\psi(i = 1,2,3)$ *are Lipschitz continuous*.s

Proof: s

$$\left|\frac{\partial\psi_1}{\partial y_1}\right| = 0, \left|\frac{\partial\psi_1}{\partial y_2}\right| = 0, \left|\frac{\partial\psi_1}{\partial y_3}\right| = 0 \quad , \quad \left|\frac{\partial\psi_2}{\partial y_1}\right| = 0, \left|\frac{\partial\psi_2}{\partial y_2}\right| = 0, \left|\frac{\partial\psi_2}{\partial y_3}\right| = 1 \quad ,$$

$$\left|\frac{\partial\psi_3}{\partial y_1}\right| \le \left|\frac{y_3}{2n\left(1 + p_{eo}\exp\left(\dfrac{(1+\alpha)y_2}{n}\right)\right)}\right|$$

$$\left|\frac{\partial\psi_3}{\partial y_2}\right| \le \left|\frac{-\left\{\begin{array}{l}\left(1 + p_{eo}e^{\frac{(1+\alpha)y_2}{n}}\right)\left((1+\alpha)\Gamma_d e^{(1+\alpha)y_2} + \dfrac{(1+\alpha)^2 P_{eo}}{n^2} e^{\frac{(1+\alpha)y_2}{n}}(y_3)^2\right) \\ \quad -\left(\Gamma_d e^{(1+\alpha)y_2} + \Gamma_r + \dfrac{(1+\alpha)P_{eo}}{n} e^{\frac{(1+\alpha)y_2}{n}}(y_3)^2\right)\left(\dfrac{(1+\alpha)P_{eo}}{n} e^{\frac{(1+\alpha)y_2}{n}}\right)\end{array}\right\}}{\left(1 + P_{eo}e^{\frac{(1+\alpha)y_2}{n}}\right)^2}\right|$$

$$\left|\frac{\partial\psi_3}{\partial y_3}\right| \le \left| -\frac{-\dfrac{y_1}{2} + 2(1+\alpha)p_{eo}\exp\left(\dfrac{(1+\alpha)y_2}{n}\right)(y_3)}{n\left(1+p_{eo}\exp\left(\dfrac{(1+\alpha)y_2}{n}\right)\right)} \right|$$

$\dfrac{\partial\psi_i}{\partial y_j}$, $i, j = 1,2,3$ are bounded since there exists a Lipschitz constant $K > 0$, such that $\left|\dfrac{\partial\psi_i}{\partial y_j}\right| \le K$, $i, j = 1,\ 2,\ 3$.

Hence $\psi_i(y_1, y_2, y_3)$, $i = 1,2,3$ are Lipschitz continuous and so (10) satisfying (11) is Lipschitz continuous

Proof of theorem 1: The existence of Lipschitz constant in the proof of theorem 2 implies the existence of unique solution of problem (10) which satisfies (11). And this implies the existence of unique solution of problem (6) satisfying the conditions (7).

Case 2: $P_{eo}e^{\frac{(1+\alpha)\theta}{n}} << 1,$

The equation (4) becomes

$$\frac{d^2\varphi}{d\eta^2} - \frac{\eta}{2}\frac{d\varphi}{d\eta} + \Gamma_d\, e^{(1+\alpha)\varphi} + \Gamma_r = 0 \qquad (12)$$

satisfying the condition (7)

Numerical Computation

Here we provide the numerical solution of equation (12) by using \Runge-Kutta Shooting method.

The equation (12) is required to be resolved into a system of equations.

Recall that
$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \eta \\ \varphi \\ \varphi' \end{pmatrix} \qquad (13)$$

Then

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 1 \\ y_3 \\ -\left[-\dfrac{y_1 y_3}{2} + \Gamma_d\ \exp((1+\alpha)\ y_2) + \Gamma_r\right] \end{pmatrix} \qquad (14)$$

satisfying the initial conditions

$$\begin{pmatrix} y_1(-1) \\ y_2(-1) \\ y_3(-1) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -\lambda_z \end{pmatrix} \quad (15)$$

Here a computer programme is written in Pascal language to solve (14) together with the initial conditions (15) of which λ_z is guessed such that the boundary condition $y_2(1) = 0$. The numerical values obtained are presented as Figure 1 below.

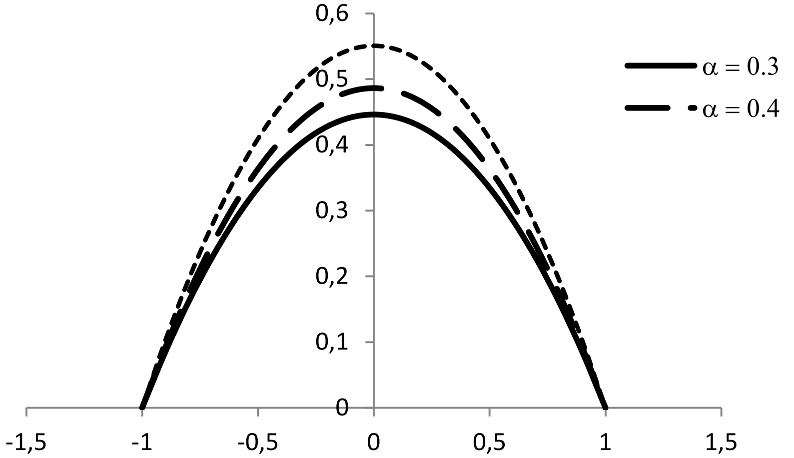

Figure 1: Temperature Profile for various α

Conclusion

We have considered a model of non-dimensional thermal conductivity equation governing the generation of heat by GD and the decay of radioactive elements in the thermal explosion in the interior of the earth. This model was first considered by Popoola (2011) when Γ_r *and* Γ_d were taken as constant. This paper revisited the model and took Γ_r *and* Γ_d as time dependent factors.

The paper also established the criteria for the existence of similarity solution. The proofs of theorems 1 and 2 showed that the problem has a unique solution and the model therefore represents a physical problem. Figure 1 showed that the activation energy ratio has appreciable effects on temperature of the reaction.

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