TOPOLOGICAL INFLUENCE FROM DISTANT FIELDS ON TWO-DIMENSIONAL QUANTUM SYSTEMS

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Abstract

A quantum system that lies nearby a magnetic or time-varying electric field region, and that is under periodic boundary conditions parallel to the interface, is shown to exhibit a hidden Aharonov-Bohm effect (magnetic or electric), caused by fluxes that are not enclosed by, but are merely neighboring to our system – its origin being the absence of magnetic monopoles in 3D space (with corresponding spacetime generalizations). Novel possibilities then arise, where a field-free system can be dramatically affected by manipulating fields in an adjacent or even distant land, provided that these nearby fluxes are not quantized (i.e. they are fractional or irrational parts of the flux quantum). Topological effects (such as Quantum Hall types of behaviors) can therefore be induced from outside our system (that is always field-free and can even reside in simply-connected space). Potential novel applications are outlined, and exotic consequences in solid state physics are pointed out (i.e. the violation of Bloch theorem in a field-free quantum periodic system), while formal analogies with certain high energy physics phenomena and with some rather unexplored areas in mechanics and thermodynamics are noted.

Keywords: Aharonov-Bohm, Gauge Transformations, Dirac monopoles, Quantum Hall Effect, Laughlin argument

Introduction

The well-known Aharonov-Bohm (AB) effect[1] occurs in systems with multiply-connected topology: the system under consideration always has to surround an enclosed and inaccessible (magnetic or spacetime-electric) flux. We here show (by providing specific physical examples) that it is possible for planar systems, with an apparently simply-connected topology, to exhibit a similar dynamic effect, but caused by nearby (rather than enclosed) or even distant (and inaccessible) fields (and corresponding fluxes), something with potential revolutionary applications – the behavior of the system not being determined by local physical laws, but mainly by nonlocal influences of fields imposed on a neighboring land.

The above has a deep gauge character, as will be shown in this paper, and it has apparently been overlooked in numerous works on extended solid state systems with inhomogeneous magnetic fields (with either conventional (parabolic) or Dirac energy dispersions), possibly because it is plagued with a gauge ambiguity. The origin of this annoying feature (the ambiguity) is here explored in detail, and it is given a full mathematical and physical explanation. It is also suggested how it can be theoretically removed (by enforcing its elimination and studying its consequences), its removal leading to macroscopic quantizations and to certain well-known properties of a topological origin (Dirac quantized magnetic monopoles, integer quantum Hall effect, quantized magnetoelectric phenomena in topological insulator surfaces). The focus of this paper is, however, the demonstration that there may occur experimental conditions (clarified here) when the transformations leading to this gauge-proximity effect or remote influence of fields do not really suffer from any ambiguity; this is due to real change in physics of a companion system in higher dimensionality that transfers momenta to our lower-dimensionality system, as will be shown with a singular gauge transformation argument (that will be different from the standard singular gauge transformation underlying the usual AB effect). Under such conditions, these proximity or remote effects are then real (experimentally realizable) and lead to the remarkable possibility of inducing topological phenomena from outside our system (which is always without fields and can even reside in simply-connected space). Specific procedures are then proposed to experimentally detect such types of nonlocal dynamical effects and exploit them for novel applications, while general consequences in solid state physics are pointed out (such as the first example of a planar field-free quantum periodic system that actually violates Bloch's theorem, this happening due to the hidden AB effect (i.e. the presence of an extra gauge field on our system that violates the standard Bloch theory) caused by the 3D companion). We also mention in passing some formal connections to certain high energy physics phenomena (θ-vacua, and some types of Gribov ambiguities), and similarities to some other, rather unexplored, areas that have attracted recent interest.

A bit of Zooming on the results: The deep reason behind the above effects is shown to be the well-known absence of magnetic monopoles[2] in higher dimensionality (3D) and corresponding generalizations in Minkowski space (whenever time-dependent fluxes are involved). These effects are here rigorously shown to exist and to affect numerous results in the literature (mostly on extended solid state systems with piecewise-continuous fields) if quantum coherence parallel to the interfaces is taken into account (through the standard imposition of periodic boundary conditions, as usually done in solid state physics). The already noted gauge ambiguity in the plane is actually due to the richer physics of the companion system in 3D that reduces to our 2D system in an appropriate limit. Under such a limiting procedure (and under certain experimental conditions) there are cases, as already mentioned, when such ambiguity is not present an apparently innocent gauge transformation in the plane (that is responsible for the gauge ambiguity) actually corresponds to real change in physics, due to nonequivalent displacements of the 3D companion relative to our 2D system (but with all of these displacements producing the same result on our lower-dimensionality system in the proper limit). This is shown to be a singular gauge transformation in 3D, namely one with a multiply-valued gauge function (but, as noted, different from the standard AB one), and it involves the above noted physical momentum transfers to our remote lower-dimensionality system, with all the physical consequences of a genuine nonlocal effect. Such type of gauge-nonlocal influence can then have important applications to extended systems that reside nearby time-dependent electric fields, or even nearby adiabatically varying magnetic fields (varying in their intensity or in their placement and adiabatic movement in 3D space), with fractional or irrational fluxes. This leads to the possibility of manufacture of interesting quantum devices that exploit the above proximity effects (i.e. a distant influence of spacetime electric fluxes) in order to induce topological phenomena from outside the system the simplest example being an electric flux-driven charge pumping in a modification of the well-known Laughlin's gauge argument[3] that is usually invoked for the explanation of the Integer Quantum Hall Effect (IQHE). By analyzing the 3D companion system we show that the above proximity effects are not only real (i.e. they can be realized experimentally), but they can also serve as an easier experimental detection of AB effects (in a simply-connected system and without enclosed fluxes (hence with lesser magnetic leakage problems)), and they can also lead to already mentioned exotic possibilities. We propose specific ways through which an experimentalist can measure effects related to the above, hinting at expected behaviors not only in a conventional 2D solid state system (i.e. with parabolic energy spectrum), but also in graphene and topological insulator surfaces (examples of quantum systems with linear low-energy spectrum).

However, in a strict planar world, with complete lack of information on the 3D companion, the above mentioned ambiguity may indeed show up (actually reflecting our ignorance of the properties of the higher dimensional companion system). This ambiguity can then be theoretically removed when certain adjacent fluxes are forced to be properly quantized; this immediately suggests a natural way to eliminate the artificial effect for confined systems, and we propose this (enforcement of elimination of the ambiguity, through quantization of nearby fluxes) as a criterion of proper behavior. Although this is not the main focus of the present paper, we argue that this has direct applicability even to cases when (effective) magnetic monopoles are present; the same criterion then directly leads to the quantization of certain macroscopic quantities, and this in turn leads to topological quantization of charge and response functions in a wide range of systems of current interest without further gauge considerations. Examples include the already noted Dirac quantization of magnetic monopoles[2], and – by additionally invoking axion electrodynamics[4] – the integral quantization of Hall conductance in conventional 2D Quantum Hall systems, and also the half-quantization of the recently proposed quantized magnetoelectric phenomena in surface-gapped 3D time-reversal-symmetric topological insulators (basically reflecting the Witten effect[5]). Finally, connections are noted with certain high-energy physics phenomena that seem to have a formal similarity (the already noted θ-vacuum sectors, and some types of Gribov ambiguities), as well as with certain areas in mechanics and in thermodynamics that are still underexplored. It may also be of interest to solid state physics that a mapping is also possible to general spin-related phenomena, through boosts to properly moving frames, providing the possibility of studying nontrivial spin-physics by starting from purely orbital considerations – although a serious look at spin-related phenomena (including spin-orbit interactions) in this new framework is reserved for a future note.

The simplest (static and magnetic) example

Consider a flat rectangle (strip) of horizontal length L in the (xy)-plane with periodic boundary conditions along L (in the x-direction), that consists of two adjacent (up and down in the y-direction) parts, again strips of length L, the one on top being empty of fields or scalar potentials (the white area) and the one at the bottom penetrated by a perpendicular magnetic field B (the dark area). Let us start with a static and uniform B, and let us first consider a nonrelativistic quantum particle (of mass m and charge e) that moves **only** inside the upper white area; i.e. we make sure that the two areas are separated by an appropriately infinite scalar potential wall, so that the lower dark (magnetic) area is totally inaccessible to the particle. Let us then set the origin $y = 0$ at the bottom of the dark area (i.e. take $(0,0)$ at the bottom left corner of the dark (magnetic) strip), the separating wall being at $y = d_1$, and the top of the white area being at $y = d_2$ (which, for simplicity, we also consider to be impenetrable). The particle is therefore confined in the y-direction by the walls at $y = d_1$ and $y = d_2$, with periodic boundary conditions (**pbc**) in the x-direction, and feels no magnetic field – the field B being only in the adjacent dark forbidden area, that lies below the particle's white strip. The usual procedure to solve this rather trivial problem, especially for the $B = 0$ case, would be to work in the gauge $\mathbf{A} = 0$ everywhere inside the white region: eigenfunctions are then of the form $\psi(x,y) \propto e^{ik_x x}\sin k_y(y - d_1)$ (with $k_y = \frac{n_y \pi}{d}$, $n_y = 1, 2, \ldots$ and $d = d_2 - d_1$, and with $k_x = \sqrt{\frac{2m\left(E - \frac{\hbar^2 k_y^2}{2m}\right)}{\hbar^2}}$ being quantized as $k_x = \frac{2\pi}{L} n_x$ ($n_x = 0, \pm 1, \pm 2, \ldots$)), with the associated energies being therefore $E_{n_x,n_y}=\frac{\hbar^2}{2m}\frac{\pi^2 n_y^2}{d^2}+\frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2 n_x^2$. Let us now include the nonzero magnetic field B (that is always inside the dark area only) by using a generalization of the Landau gauge, with the origin being as noted above, namely $\mathbf{A}=-yB\hat{e}_x$ for $0<y<d_1$, and $\mathbf{A}=-d_1B\hat{e}_x+A_0\hat{e}_x$ for $d_1<y<d_2$; this gauge choice indeed satisfies that $\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}$ is B inside (and zero outside) the dark region, and $\mathbf{A}$ is continuous at the separating wall (at $y=d_1$). Note that the particle in the white area now feels a nonzero (although uniform) vector potential, that makes wavefunctions formally pick up an extra phase factor $e^{i\frac{e}{\hbar c}\int \mathbf{A}\cdot d\mathbf{x}}$ (through a gauge transformation mapping trick, starting from $\mathbf{A}=0$), so that we now have $\psi(x,y)\sim e^{ik_x x}e^{i\frac{e}{\hbar c}A_0 x}\sin\frac{n_y\pi}{d}(y-d_1)$. By then imposing the **pbc** in the x -direction, we obtain $e^{i\left(k_x-\frac{e}{\hbar c}A_0\right)L}=1$. From this, we can determine the new quantized values of k_x and then the energy spectrum, which finally turns out to be

$$E_{n_x,n_y}=\frac{\hbar^2}{2m}\frac{\pi^2 n_y^2}{d^2}+\frac{\hbar^2}{2m}\left(\frac{2\pi}{L}\right)^2(n_x-f)^2,$$

with $f=\frac{\Phi_{dark}}{\Phi_0}$, where $\Phi_{dark}=Bd_1L$ is the total flux through the dark area and $\Phi_0=\frac{hc}{e}$ is the flux quantum. These allowed energies are actually periodic with respect to Φ_{dark} (with period Φ_0) – as can be seen if, for a given Φ_{dark} , proper shifting of the integers n_x is made – and whenever Φ_{dark} happens to be an integral multiple of Φ_0 , the global spectrum is equivalent to that corresponding to the absence of the adjacent B (i.e. to $\Phi_{dark}=0$) , reducing to the one with $\mathbf{A}=0$ derived earlier).

The key observation is that, although the particle will never enter the dark area, its energy spectrum, and from this other measurable quantities (i.e. global electric current $J=-c\frac{\partial E}{\partial \Phi}$) are seen to be affected by the adjacent (forbidden) magnetic field – a type of proximity field influence, and not the usual AB effect, since the magnetic flux is **not enclosed** by the region where the particle resides, but is only adjacent to it. If the origin of our coordinate system were chosen anywhere below the dark floor, the above result would seem to be origin-independent. If however we chose the origin to be, e.g., at the wall separating the two areas, then this effect would go away. (And note that in flat space, change of origin is equivalent to a gauge transformation – see further on this later below). We observe therefore a gauge ambiguity. Hence one may well say that it cannot be a real physical effect; the theory however *does predict* such an artificial effect as a direct consequence. What is the reason behind it or what is its deeper origin? And, most importantly, is it ever possible to make any use of it experimentally? We shall see that this can be given an affirmative answer, under certain conditions.

A direct first understanding of why an adjacent region can affect our system can immediately be provided by the appearance of nonlocal terms in a gauge function, in the generalized theory of refs [6,7], something that occurs whenever, in the standardsolutions (Dirac phases) with integrals of vector or scalar potentials, these integrals have to pass through regions with nonvanishing fields (although the final point of the integrals is always outside these fields). However, even in the standard framework of the usual Dirac phase factors, the results are the same (as derived above), and the deeper origin of the above proximity field influence in flat space can be revealed through 3D folding (compatible with the pbc in the x -direction): we show in what follows that the above effect is actually due to the well-known absence of magnetic monopoles anywhere in the embedding 3D space. Indeed, by folding in the x-direction in order to form a cylinder (by gluing the opposite vertical sides), the above gauge, now written in cylindrical coordinates, has only azimuthal component, and it is $\mathbf{A} = (0, -zB, 0)$ in the dark and $\mathbf{A} = (0, -d_1 B, 0)$ in the white area, with B always denoting the magnitude of the locally perpendicular field (at every point of the now dark folded strip) that has now become the *radial* component of a larger magnetic field $\mathbf{B}$ in 3D space. It is crucial then to note that this gauge choice leads in 3D to the *additional appearance of a nonzero* B_z (component of this larger field $\mathbf{B}$ parallel to the cylinder's z-axis) that is *inhomogeneous* (generally ρ- and z-dependent). Indeed, straightforward calculation of the total field $\mathbf{B}$ produced by the above form of $\mathbf{A}$ (see next Section) leads to $B_z = -\frac{zB}{\rho}$ in the dark area, and $B_z = -\frac{d_1 B}{\rho}$ in the white area. This inhomogeneous B_z in all 3D space is exactly what is needed to give a flux (of this B_z) through the top (say at height z_2) and the bottom (say at height z_1) of any cylinder (of height $z_2 - z_1$) that overall *cancels out* the radial flux (of B_ρ) that goes through its curved cylindrical side-surface (with this B_ρ partly being B and partly zero in the corresponding dark and white regions that now lie folded on the surface). And the flux of B_z through the top is also identical to the value of a horizontal closed integral of the corresponding $\mathbf{A}$ at height z_2 – this way directly demonstrating that we now have the standard AB effect operating (since the B_z-flux *is* now *enclosed* by the particle's region). What we see here is simply the well-known fact that the total flux passing through the entire closed cylindrical surface is indeed zero (as demanded by the volume integral of $\nabla \cdot \mathbf{B} = 0$ inside the whole cylinder). Hence, in the case of $z_2 = z$ being in the white region, and $z_1 = 0$ (so that the flux-contribution from the bottom at $z_1 = 0$ is vanishing), the proximity field influence at height z inside the white area is a hidden (or indirect) AB effect, due to the enclosed flux of B_z that is automatically created, which in turn is equal (up to a sign) to the dark flux, i.e. the flux of the radial B through the entire dark folded strip, due to the above cancellation. This way **the dark strip affects indirectly the adjacent white region (**through the companion system in 3D, i.e. through the automatic formation of the appropriate 3D magnetic field $\mathbf{B}$ that must satisfy all the above (and everything that follows, see next Section)**)**.

Flux cancellations and physical explanation

Let us first briefly confirm the above mathematical results and then provide a physical understanding. Recall that $(\nabla \times \mathbf{A})_\rho = \frac{1}{\rho}\left(\frac{\partial A_z}{\partial \varphi} - \frac{\partial(\rho A_\varphi)}{\partial z}\right)$, $(\nabla \times \mathbf{A})_\varphi = \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)$ and $(\nabla \times \mathbf{A})_z = \frac{1}{\rho}\left(\frac{\partial(\rho A_\varphi)}{\partial \rho} - \frac{\partial A_\rho}{\partial \varphi}\right)$. For our gauge $\mathbf{A} = (0, -zB, 0)$ in the dark area and $\mathbf{A} = (0, -d_1 B, 0)$ in the white area, these lead to: $B_\rho = \frac{1}{\rho}\left(-\frac{\partial}{\partial z}(-\rho z B)\right) = B$ in the dark area, and $B_\rho = \frac{1}{\rho}\left(-\frac{\partial}{\partial z}(-\rho d_1 B)\right) = 0$ in the white area as required; we also obtain $B_\varphi = 0$ in both areas, and finally $B_z = \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(-\rho z B)\right) = -\frac{zB}{\rho}$ in the dark area, and $B_z = \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(-\rho d_1 B)\right) = -\frac{d_1 B}{\rho}$ in the white area, as claimed above. In order to make the above mentioned cancellations easily visible, take the special choice $z_1 = 0$ (the bottom of the cylinder being at the origin (at the bottom of the dark strip)) and $z_2 = z$ (the top of the cylinder, lying either (a) inside the dark or (b) inside the white area); the curved side-surface then consists of either (a) just a lower part of the dark strip or (b) the entire folded dark strip, together with a lower part of the white folded area. Then indeed, the flux of B_z through the ceiling is $\iint B_z \rho d\rho d\varphi = -zB2\pi R$ if z is inside the dark area, or $-d_1 B2\pi R$ (hence a constant) if z lies inside the white area; and we see that, in either case, it indeed cancels out the radial B -flux (which is $zB2\pi R$ in the dark area and the constant $d_1 B2\pi R$ in the white area, either of which can also be determined by use of the proper B_ρ as given above). And the flux of B_z through the top is also identical to the value of a closed integral of the corresponding $\mathbf{A}$ around the cylinder (which is $\oint A_\varphi R d\varphi = -zB2\pi R$ or $-d_1 B2\pi R$) as expected, this way clearly demonstrating that the above cancellation is actually due to the standard AB effect (due to the B_z -flux that *is enclosed* by the particle's region).

After this mathematical confirmation, and in an attempt to provide a better physical understanding and also seek an experimental realization, let us momentarily turn to a slightly different gauge, namely $\mathbf{A} = \left(0, -\frac{zBR}{\rho}, 0\right)$ in the dark area (a gauge discussed earlier[8], together with an actual realistic current distribution $\mathbf{J}$ that produces it) and $\mathbf{A} = \left(0, -\frac{d_1 BR}{\rho}, 0\right)$ in the white area (all this being compatible with our own gauge for $\rho \to R$ the radius of the cylinder). This gauge, produced by a $\mathbf{J} \propto -BRz\hat{e}_\varphi$, can be shown to lead to similar cancellations and a similar conclusion of influence of remote fields as with our initial choice. But, more importantly, in both gauges, the value of B_z changes with the location of origin; i.e. in our first choice of gauge, a direct calculation as above (but with shifted origin) now gives $B_z = -\frac{(z-z_0)B}{\rho}$ for z in the dark area, and $B_z = -\frac{(d_1-z_0)B}{\rho}$ for z in the white area, and the presence of the arbitrary constant z_0 in the results can be seen as the actual source of the gauge ambiguity noted earlier. It has to do with a different flux balance (in the overall cancellation) between the top, the bottom, and the side-surface of any considered cylinder, and this will generally give *an origin-dependent fluxthrough the top* – hence leading to a z_0 -dependent AB influence at height z , and therefore a y_0 -dependent proximity influence in the initial flat system. Note that, in both 1st and 2nd choice of gauge, the point z_0 is always the point (height) where B_z (or $\mathbf{J}$) vanishes (see also ref.[9], fig.3, for a related (but simpler) system, where a similar B_z , with a vanishing point, also shows up) – these observations being important for our later discussion (Section V) on a relevant experimental setup.

In spite of the above peculiarity however (namely, the extra appearance in higher dimensionality of a B_z that actually has a vanishing point with a completely arbitrary location), the crucial property to note is that, when the plane is flat (i.e. in the limit $R \to \infty$),**the above B_z always goes to zero** on the surface (for any finite z), because of its $1/\rho$ - dependence (whereas for the 2nd gauge it is exactly zero on the cylinder surface because of a delta function centered on the axis). Although B_z is zero in the planar system, we see, however, that the memory of a finite enclosed flux in infinite 3D space remains, and it is this that actually causes the proximity field influence in the planar case. It is as if the cylinder axis has moved to infinity in such a way that B_z through the infinite space gives the same flux as for the folded system, namely $B_z \to 0$, but $\rho \to \infty$ in such a way that their product is either $-yB$ (dark area) or $-d_1 B$ (white area), which, in fact, *are* the correct values of $\mathbf{A}$ for our planar system, but now derived by a limiting procedure. It is also interesting to emphasize that the nonlocal terms appearing in the gauge functions of the theory of refs [6,7] for 2D static magnetic cases confirm (or, better, *contain*) this type of proximity influence **directly in flat 2D space**, without the need of any folding (or unfolding) or other limiting considerations – a point of importance that we are planning to get back to in the near future.

Regarding a possible connection of the above y_0 -ambiguity to *real* physics, note that, mathematically, a gauge transformation in the planar problem (upon displacement of the origin y_0) *is* an ordinary gauge transformation (the gauge function is $\Lambda = -\delta\Phi \frac{x}{L}$ (with $\delta\Phi$ being the change of flux that corresponds to a change of vector potential $\delta\mathbf{A}$, a quantity that will appear below to be involved in a momentum transfer) and is indeed a smooth single-valued function everywhere on the plane); when however we fold into a cylinder, the corresponding Λ turns out to be $\Lambda = -z_0 BR\varphi$, that is basically identical to the above planar Λ, but is now multiply-valued (it has the usual discontinuity with respect to the azimuthal variable φ appearing in all magnetic AB types of phenomena in a cylindrically symmetric configuration). Hence in 3D the change of origin z_0 is not an ordinary gauge transformation but a singular one, and it is expected to reflect real physics (or, more accurately, a real change in physics between behaviors before and after the transformation, to be clarified below). The situation is similar to the standard AB effect (that introduces real difference in physics, compared to a particle free of potentials) but *not identical*: here the effect is *defined by the surface radial* B in the dark region (and by the choice $C = 0$ in a more general gauge, see below) and there is no additional arbitrary flux allowed to pass through the empty space – the one that appears in our problem (the flux of B_z) having shown up *automatically* due to the absence of magnetic monopoles ; in other words, and now mathematically speaking, in an otherwise legitimate choice of gauge $\mathbf{A} = (-yB + C)\hat{\mathbf{x}}$ etc., we have always used $C = 0$, hence not allowing the usual AB effect (i.e. not allowing an extra arbitrary magnetic flux that one could always add inside the cylinder without affecting the fields on the surface – the point being that the effect we present here *appears by itself* due to the surface radial B -field, and it is not caused by additional and arbitrary flux-insertions). Furthermore, and now physically speaking, the z_0 -freedom has to do with the *different (infinite in number) arrangements* of the total magnetic field (in 3D space) that *all produce the same 2D values of fields on the side surface* (namely the same radial field component, either zero or B in the corresponding strips) and therefore produce the same physical field-arrangement of our initial planar system. Indeed, note that the formal appearance of $(z - z_0)$ in B_z, actually reduces the above mentioned ambiguity to an ambiguity with respect to *displacements of the point where* B_z *vanishes*. And note that there is a great arbitrariness in placing the point of vanishing B_z somewhere in 3D space, although the 2D system does not know of all this freedom -- it only senses the radial field, which is *always* (for *any* of these $\mathbf{B}$ -constructions) the same – in our case it is B in the dark area and zero in the white. And then, any such change of location of the vanishing point z_0 involves relative displacements of the total $\mathbf{B}$ -field in 3D space (relative to the cylinder), and this must be the source of *momentum transfer* to the particle on the surface. Indeed, such momentum transfer (integrated over infinite time) turns out to be equal to $q\delta\mathbf{A}/c$ (as can be shown by following lines of reasoning similar to those of ref.[10]) and gives an explanation for the physical origin of the extra phases (of AB type) picked up by the particle's wavefunctions upon change of z_0 . Summarizing, the crucial element is that our original planar system, with the **pbc**, is an effectively compact system (and can always be viewed as the $R \to \infty$ limit of a compact cylinder), and due to the compactification, the gauge transformations are not so innocent (they are actually singular, and hide real physics), the nontrivial effects having as origin the above noted displacement of the $\mathbf{B}$ -field in 3D space and the associated momentum transfers to our surface-particle. [Note also that, although in the planar system B_z *vanishes everywhere*, the special point z_0 (or now y_0) has already been identified (as the unique point of local vanishing of B_z in the 3D companion system) *before taking the limit* – something that will be of relevance to the experimental discussion later in Section V.]

Effects of the above type are actually implicit in carbon nanotubes[11] (with metallic nanotubes subject to the above pbc), and also have immediate applicability to planar graphene (with no curvature)[12], although the above ambiguity has not to our knowledge been discussed (or exploited) – see however later below (Section VI) for our own suggestions on what to expect in such proximity measurements in graphene and topological insulator surfaces.

Consequenses on other works and Generalization

Let us first briefly point out some consequences of the above effects on previous works, and discuss certain important generalizations, together with issues of experimental relevance (on how i.e. these proximity influences could be detected in the laboratory).

(A) The above types of effects seem to also appear in connection with the concept of effective scalar potential that has been extensively used in previous works (both on conventional systems[13,14] and on Dirac materials[15,16]) and in cases that the field is *accessible* to the particle (although this is not the focus of the present work – the case of forbidden fields making our proximity effect more striking (or physically unexpected)). Indeed, the above noted gauge ambiguity shows up as a *gauge-dependence of the effective scalar potential* (that seems to have also escaped notice), and it seems to affect even the *qualitative form* of this potential in the white area (see i.e. fig. 1(b) of the first of ref.[13]) – this form depending on the combination of d_1 and the sign of k_x (see below) – bringing about important changes in measurable quantities in either conventional or Dirac systems. All examples in the literature consist of systems with magnetic strips or barriers that have been discussed (for accessible fields) by matching methods. For parabolic energies the effective potential turns out to be $V_{eff}(y) = \frac{(\hbar k_x - \frac{e}{c} A_x(y))^2}{2m}$, and in the white area $A_x(y)$ is a constant $-d_1 B$, whose value is d_1 -dependent, and it is matched with the form of V_{eff} as this comes from inside the field at the interface; inside the field we have $V_{eff}(y) = \frac{1}{2} m \omega_c^2 (y - y_0)^2$ with $y_0 = k_x \frac{\hbar c}{eB}$, and it is clear that if d_1 is not an integral multiple of y_0 , then we have nontrivial consequences on the form of the potential (and therefore of the wavefunctions) outside the magnetic region (whereas if $d_1 = N y_0$, with N integer, then Φ_{dark} is quantized and there is no new effect). In the case of Dirac materials, by using the Dirac Hamiltonian $H = v_f \boldsymbol{\sigma} \cdot \boldsymbol{\pi}$ (with $\boldsymbol{\pi} = \mathbf{p} - \frac{e}{c}\mathbf{A}$ the kinematic momentum) and with ansatz $\psi_i(x, y) = \psi_i(y) e^{i k_x x}$ (with $i = 1, 2$ denoting the components of a spinor) it turns out that for the white area we have to solve a system of Schrödinger-like equations, namely

$\left(-\hbar^2\frac{\partial^2}{\partial y^2}+\left(\hbar k_x+\frac{e}{c}A_x(y)\right)^2\right)\psi_{1,2}(x,y)=\frac{E^2}{v_f^2}\psi_{1,2}(x,y)$, and we clearly see a similar effect (and gauge-dependence) as in the nonrelativistic system (the detailed solution will be given elsewhere[17]). In case that the dark strip has no integrally-quantized flux, the solution is again not equivalent to the case of flux-absence. Once again, at the bottom of this is phase-physics (and the phase-mismatch around the cylinder when Φ_{dark} is not quantized). And if we follow this method of effective scalar potential for our original striped system with the magnetic region being again *inaccessible*, then it turns out (in a quite different manner from what we did in the beginning of this paper) that the energy spectrum in the white area is identical to eq.(1), with $f=\frac{qd_1B}{ch/L}$ which is Φ_{dark}/Φ_0, in agreement with our gauge transformation mapping technique. Hence the use of the effective scalar potential method — and the solution based on matching conditions in a direction *transverse* to the interface — seems to lead to the same results as those of a phase-mismatch analysis *parallel* to the interface.

In a similar vein, systems such as a striped one discussed in Zygelman's recent work[18] are also expected to be affected — if we impose periodic boundary conditions parallel to the strip — whenever the flux of the strip is *not quantized* (and it is easier to see this if we take the strip to be a delta function). A detailed solution will be given elsewhere[17] with the direct use of the concept of *pseudomomentum* and its generalization to inhomogeneous fields (and how it is affected across the interface from inside to outside the field for piecewise-continuous cases). However, note again that the focus of the present work is *not* on fields sensed by the particles, but on inaccessible fields, because it is these cases that may make the effect of nearby fluxes appear more unexpected.

(B) It is also important to note that the above folding procedure of our dark-and-white system actually generalizes Laughlin's gauge argument on a cylinder[3], where, however, the automatic appearance of the above B_z (upon folding) is, to our knowledge, rarely (if ever) discussed. And the addition of our white strip on the surface of the usual Laughlin cylinder gives nontrivial consequences **whenever the outside magnetic flux is not quantized** (see below, on effective pumping and IQHE conditions induced from the outside).

In the standard Laughlin's argument, with a radial B being present *everywhere* on the cylinder's curved cylindrical surface, one can actually understand the well-known translational symmetry breaking[19] — where the equilibrium positions of the standard Landau wavefunctions ($y_0=k_xl^2$ in planar language, with $l=\sqrt{\hbar c/eB}$ the magnetic length) become priviledged[19] — by the special consideration of this additional B_z created due to folding： as we saw, the AB flux enclosed by a horizontal circle (lying on the cylindrical surface) around the axis depends on the height z (due to the presence of the $B_z=-\frac{zB}{R}$), so that, if we want immediate wavefunction single-valuedness around the cylinder, we indeed need special z 's so that the enclosed AB flux (at that height) is quantized (in integral multiples of Φ_0). It is straightforward to see that this requirement gives immediately the priviledged z_0 's (or equivalently the above equilibrium positions y_0 's for the standard flat Landau problem in the Landau gauge). But further than that, in our generalized system, with the area of interest (where the particle resides) being only a white strip on the cylindrical surface (with no field B inside it), one finds that there are nontrivial consequences (due to remote field influence) on this white area, whenever the outside magnetic flux is not quantized. This we saw with inaccessible fields, but it seems to also occur for accessible ones as well, as we demonstrated above. In such case of non-quantized Φ_{dark} outside our white area, the wavefunction single-valuedness (or pbc along x) in the white area is *not* automatically satisfied, and it is its enforcement that leads to a modification of physical properties, hence to the remote influence of the adjacent magnetic field that we saw. A plausible question would then be: is there a remote (or proximity) influence of the IQHE type that might affect the particle, although this resides **outside** the field B (hence, equivalently, a quantum Hall type of effect in zero-field)? There is a great deal that can be said on this – i.e. in relation to magnetic edge states in the interface[20], snake states[21] etc. to be discussed in a more focused paper, the main conclusion for now being that we must have nontrivial dissipationless edge currents in the interface that, in any case, are expected, as the persistent currents associated with the hidden AB effect, being therefore proportional to $\partial/\partial y_0$; but even without details, we will point out as certainly true that one can generate (or simulate) IQHE conditions on our system (always a white area, with no B) with a pulsed outside electric field – rather than the static field case discussed in the beginning – which, due to its time-dependence, can induce IQHE type of effects inside our field-free system (a case now involving remote electric fluxes in spacetime). An even simpler way is our original example of a magnetic field B in the dark area, which however is not static but slowly (adiabatically) changing with time, or, alternatively, a fixed B while our origin y_0 is being displaced slowly (and transversely to the interface) between two values *that correspond to a change of flux in the dark area equal to* Φ_0 (this would then define a cycle). This way one can achieve charge pumping (with slow variation of B or of y_0 or proper combination of both) as in the case of Laughlin's cylinder[3], replacing the much harder to build externally applied varying *enclosed* AB flux. After a cycle, there must be an integer number of electrons transported from one side of the system to the other (along the y -direction), a well-known topological quantum effect (the so-called adiabatic particle transport) due to Thouless[22]. Or one can use other more sophisticated types of procedures based on nonlocal terms in refs [6,7] involving general t -dependent electric fields and electric scalar potentials. Summarizing, it seems that, in a number of different ways, one can induce conditions of, at least, topological (quantized) pumping of some quantity, resulting from manipulations from *outside* of our system, and, in fact, in ways that are expected to *respect relativistic causality*, as shown in detail in refs [6,7].

(C) One should note that all the phenomena predicted here should be observable, independent of our (or any other) analysis of the z_0 -ambiguity. One can give z_0 an absolute meaning (for a particular *cylindrical* system in the laboratory): it is the point in the 3D folded system at which the z -component B_z of the total 3D magnetic field $\mathbf{B}$ (or its source, the current density $\mathbf{J}$) vanishes. We can therefore determine this point z_0 in our 3D setup (see i.e. in fig.3 of ref.[9] the point where the magnetic lines are curved in opposite directions), and then be careful to place our system of interest (i.e. a strip with no field, exhibitting quantum coherence parallel to the interface with the dark magnetic region) in a manner so that its basis (namely the interface itself) is *displaced* (by a small distance d_1) with respect to z_0 . Then, if this distance d_1 is such that the outside magnetic flux is not quantized, then the above effects (a proximity influence of this flux on our white strip) should be present and measurable. [If they are not ever found, then something is wrong with standard quantum theory and/or (classical) electromagnetism.] And, as shown earlier by a limiting procedure, these proximity influences must survive even after the system becomes flat. However, a question arises about cases when we *start* with a *strictly flat* system, with no knowledge of the location of the B_z -vanishing point of a corresponding 3D companion. For such cases, we will argue that we have two options to consider： for the 1st, see next Section (where it is shown that a possibility still remains to have a nonlocal proximity effect with no ambiguity), and the 2nd is the case of actually having the y_0 -ambiguity, which is now physically unacceptable, and thenour criterion of proper behavior (noted earlier) *must be enforced*. This enforcement of elimination of the ambiguity then seems to lead to **(a)** topological physics (manifested as quantization of certain quantities, such as magnetic charge and response functions), as well as to **(b)** connections and formal analogies with other physics areas. Indeed, **(a)** recall that, in all the above, essential use was made of the nonexistence of magnetic monopoles in 3D (the $\nabla\cdot\mathbf{B}=0$ law). But what if we had assumed that magnetic monopoles exist? Our simplest finding on thisis that imposition of our criterion of proper behavior (forced elimination of the z_0 -ambiguity) leads to quantization of fluxes external to the white system, so that, in the limit that our white system shrinks to zero, the nonlocal term of [6,7] can serve as a *probe of quantization* of the flux through the outside magnetic regions; and the enforced quantization of the nonlocal term leads, in turn, to the quantization of magnetic charge according to the Dirac condition[2], and more generally, to the quantization of other macroscopic quantities, that are related to quantized magnetoelectric effects in an axion electrodynamical consideration[4] (see further below, Section VII). In particular, our above criterion seems to nicely complement the recent proof of the 2π -periodicity of the axionic action[23] by providing a justification of the quantizations of certain separate 2D fluxes (one in 2D space and one in 1+1-D spacetime) that are crucial in the proof, justification that is not given in ref.[23]. **(b)** Apart from the above, there are much wider implications (mainly *physical*), but also relationships with other physics areas that one can see *formal* analogies with (see below brief discussion on axions, θ -vacuum sectors[24,25], Gribov copies[26], but also connections with certain open problems in mechanics[27] and in thermodynamics[28]), that certainly necessitate further investigation of an interdisciplinary character.

How to measure nonlocality in a strictly planar system

First, for a cylindrical arrangement, we have seen that the special vanishing- B_z point (z_0) is unique and identifiable, and survives in the $R\to\infty$ limit, so that the remote influences that are the focus of the present work *must survive even after the system becomes flat*; and although in the completely planar system B_z *vanishes everywhere*, we have already identified the absolute reference point z_0 (or y_0) *before the limit* (as the unique point of vanishing B_z that existed in the companion 3D system). Hence, by using this y_0 , we can achieve (or measure) all the types of proximity effects discussed above in the same way (namely, by placing our white area in a properly displaced manner with respect to this y_0).

However, for strictly planar system, when we have no knowledge of the B_z -vanishing point of a corresponding 3D companion, we argued earlier that we have two options to consider, and here we focus on the first： If we have a large-width (d_1) magnetic area, it is quite possible that, generically, this would behave as if it were produced by a corresponding long cylinder (in the usual theoretical limit $R\to\infty$) with its special vanishing- B_z point (z_0) being in the *middle* of its *finite* length; this is for symmetry reasons and due to the fact that all expressions of the fields used here (and in fact in the entire literature) are actually exact only in the case of *infinite* cylinders -- the middle of a long cylinder being therefore slightly preferred (as being the point that is more distant from both cylinder-ends, and also because, due to its symmetrical placement, it is a better representative of the infinite-cylinder theory). If this turns out to be correct, then this suggests an obvious experimental way on how to place our white area: y_0 can be taken to be in the middle of the width of the flat dark area, and then our white system must be placed as described earlier. In fact, a slightly better experimental suggestion would be to have *two* systems of interest (white areas, i.e. they could be identical graphene samples), separated by the above (inaccessible) wide magnetic region, and then make measurements (i.e. of persistent current) in one system *or* the other; the point is that, no matter where y_0 is located, *at least one* of the two systems *must* be affected by proximity (if i.e. it happens that y_0 is at the edge of one area, giving no effect on it, then the same y_0 is necessarily displaced with respect to the 2nd area; so proximity influence on the 2nd system is guaranteed, if the intermediate flux is not quantized, and we can measure nontrivial effects in this 2nd system and it is interesting to note that, if y_0 is indeed in the middle of the magnetic region, as we hoped earlier, then now, in the present setup, both systems will be affected equally). If all this does not work (meaning that there is no memory of a *unique* y_0, a remnant of the theoretical limit), then the lack of knowledge of a 3D companion is indeed complete, or equivalently this gives rise to the earlier discussed ambiguity. In such case, as already noted, our criterion of proper behavior *must be imposed*, and the consequences of this are briefly discussed in Section VII.

Predictions on Graphene and Topological Insulator

An outline of the simplest possible types of measurements (related to the gauge proximity effect presented in this work) in *conventional* systems has been given earlier in this paper (mostly on induction of IQHE-type of effects and charge pumping, all induced from outside the system). It should be stressed, as a generic feature (and as a prediction) that, even if our white area is almost empty (i.e. a single electron in empty space), we would at least expect (persistent) currents along the edge (interface between white and dark areas) this being valid for real solid state systems with both parabolic and Dirac electronic spectrum in AB configurations[29]. This was also noted for our own proximity configuration above (with the expectation that J will be proportional to y_0).

But beyond this, we here also provide our more detailed predictions of what one would expect on general grounds, if our white system is one of the two most popular nowadays topological materials, graphene or a topological insulator :*Graphene***:** proximity arrangement with a $\mathbf{B}$, would offer a controllable way (through changes of the outside $\mathbf{B}$ or of y_0) to *lift* the *orbital degeneracy* that originates from the two valleys, with consequences on persistent currents (in x -direction) and in conductance (i.e. some shifting of peaks), analogous to the ones of ref.[30]. In addition, giant magnetoresistance at room temperature is possible, due to the hidden AB interference[31]. *Topological insulators***:** By way of an example, in the proximity to an HgTe quantum well one would expect to measure helical edge states, bound states and persistent currents (with Rashba spin-orbit coupling), that would generally be affected in a manner similar to the one described in ref.[32]. On all this, we plan to return with details (and experimental suggestions on each material) in a future note.

Removal of the ambiguity

Although not the focus of the present paper, let us briefly mention the manner in which topological physics shows up upon the enforced removal of the ambiguity, and let us first consider cases where (effective) magnetic monopoles are *present*. Note that, already in the case of the Laughlin cylinder discussed earlier – with the usual in the literature practice of not any mention of the extra B_z that originates from folding of the original flat system – it is seen that the radial B in 3D space *must* be a result of a linear magnetic monopole distribution (along the z -axis) – since a purely radial field violates the $\nabla \cdot \mathbf{B} = 0$ law (as there is a nonzero net flux outwards and, therefore, magnetic monopoles must be invoked to justify it). And starting with an additionally placed extra narrow ($d \to 0$) white strip (with no field) that goes around the axis on the cylindrical surface, and imposing our criterion of proper behavior (forced elimination of the gauge ambiguity) in the limit $d \to 0$ one obtains the well-known quantization of the B -flux in the dark area, and from this it comes out that the monopole charge must also be quantized (see ref.[33] for quantitative details on how the Dirac's quantization condition comes out). By formally enforcing the elimination of this gauge ambiguity in a closed system, the nonlocal term (namely, the flux lying outside our system) can play the role of a probe of (or a detector of) quantization of macroscopic quantities (although, it should be noted, *we are merely at the level of wavefunction phases*). A plausible question then is: can such a type of argumentation be followed for other more complicated cases? We answer positively by working out some considerably more sophisticated examples (with topologically nontrivial systems), which, as has been shown recently[34,35], seem to need axion electrodynamics to describe their exotic magnetoelectric response properties. The reader is again referred to [33] that shows in detail that imposition of our criterion of proper behavior leads to quantization of the *axionic* current density $\mathbf{J}_\theta$, which in turn leads, for conventional IQHE systems to $\sigma_H =$ an integral multiple of $\frac{e^2}{h}$, and for topological insulator surfaces that are in contact with a topologically trivial medium (i.e. the vacuum) to $\sigma_H =$ an odd integral multiple of $\frac{e^2}{2h}$; the same method also leads to their quantized magnetoelectric responses, in accordance with the Witten effect[5] (see [33] for details).

Formal analogies with other areas

The wider physical implications, and/or relationships with other physics areas have also been examined in [33], where *formal* connections have been noted, among others, (**i**) with recent considerations of Berry and Shukla[27] on curl forces that are spatially confined in classical systems (while the point of observation is *outside*, in curl-free regions), (**ii**) with not yet well-studied issues of irreversibility and vorticity in thermodynamics[28], (**iii**) with extensions to spin-physics[36], and (**iv**) with certain quite esoteric issues in high energy physics, such as θ -vacuum sectors[24,25] being formally analogous to our y_0 --sectors, and the so-called Gribov problem (or Gribov ambiguity[26]); for such a claimed connection see in particular refs [37] and [38] where the existence of the Gribov phenomenon is related to the existence of inequivalent quantizations (which in our simpler problem corresponds to different y_0 -sectors), and then Gribov copies are labeled through procedures that are formally similar to ours.

Discussion and Conclusion

Even without the above generalizations, however, the simplest outcome of the present theoretical work – that it is in principle *possible* to have effects without fields, *in the simply-connected plane,* that are generated outside our system and that affect its physical properties – is remarkable, and if true, extremely important in experimental work on fundamental physics as well as in practical applications. First, the most obvious use is for an easier experimental detection of AB effects, as already noted (with considerably lesser problems of leakage of magnetic lines, compared to typical enclosed-flux arrangements). Then, the already noted possibility of violation of Bloch theorem (especially if our white (no-field) system is periodic along the interface direction) is worth emphasizing. The violation is due to the presence (on the system) of the extra vector potential (from proximity with the outside B -field), and it leads to AB-type of modifications of the translation operators that are used in the standard proof of the Bloch theorem. [It should be noted that these modifications are *not* the same as the well-known modifications of Bloch theorem in an IQHE system (such as the ones studied i.e. in [39]) with the particle being inside a field — in our case we always have $B = 0$ on the particle.] We therefore eventually expect nontrivial modifications in the form of wavefunctions; in such a case, one can first gauge away the proximity-induced $\mathbf{A}$, with the consequence of the extra appearance in the boundary conditions of a crystal momentum (parallel to the strip). And then, by adiabatically changing the special point y_0 in a direction transverse to the strip by a cycle (meaning that the corresponding change of flux is equal to Φ_0), we can have the crystal momentum moving from one edge to the other of the (parallel) Brillouin zone, and hence induce new effects (or transitions) that can lead to interesting physics, especially if electron-electron interactions are taken into account. It is also interesting, and potentially useful experimentally, that, in cases when both electrons and holes are considered, the Berry phase picked up during such a cycle seems to contain not only an AB part (as derived by Berry in the transported rigid box around an AB flux[40]), but also a term directly related to the *electric current*, similarly to what happens in an AB ring[41,42]. Finally, a periodic (or even quasiperiodic, i.e. Fibonacci) arrangement of magnetic strips (on a cylinder, or in the plane with pbc parallel to the strips), each one containing a rational flux $f\Phi_0$ (with $f = \frac{p}{q}$, p,q integers, with $p < q$), would be an interesting system to consider, with new (in)commensurability effects expected (not of the Hofstadter type[43] where we have a nonzero B -field), that will be a result of the interplay between the gauge proximity effects of the present work and the (quasi)periodicity of the structure — behavioral patterns that will be possibly useful for novel applications in intelligent devices.

Regarding all the above, it is for the experiment to give the verdict, but it is fair to say that we have provided in this work strong theoretical evidence (in fact a rigorous proof) for the existence of a proximity effect (or even remote influence of fields from a distance) that has a deeply gauge nature — something remarkable, and important at least for novel applications. And although we have focused on orbital physics, there are well-defined steps (through boosts to properly moving frames) that lead to spin-physics as well — although a generalization of the U(1) gauge character of the nonlocal effects proposed here to cases with a spin-orbit coupling (now with an SU(2) character) would have an additional importance for modern applications and, as already noted, deserves a separate note. This demonstrates that, if the above proximity effects turn out to be real, the experimental and application possibilities of exploiting them, as well as their generalizations, seem to be almost limitless.

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