CONTRIBUTION TO IMPULSIVE EQUATIONS

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Abstract
In this paper, we show the validity of the method of upper andlower solutions to obtain an existence result for a first order impulsive differential equations with variable moments.

Keywords: Impulsive differential equation, variable times, upper and lower solutions.

Introduction

In this paper we will consider the following system of differential equations with impulses at variable times:

I.

The investigation of theory of impulsive differential equations with variable moments of time is more difficult than the impulsive differential equations with fixed moments. This paper concerns the existence of solutions for the functional differential equations with impulsive effects at variable times. We consider the first order initial value problem (IVP for short):

$$y^{'}(t) = f\big(t, y(t)\big) \quad a.e \;\; t \in [0,T], \qquad t \neq \tau_k\big(y(t)\big), k = 1, \dots m$$
$$y(t^+) = I_k\big(y(t)\big) \;\; t = \tau_k\big(y(t)\big), \qquad k = 1, \dots, m \qquad (1,1)$$
$$y(t) = \varphi(t) \quad t \in [-r, 0]$$

Here $f: [0,T) \times D \to \mathbb{R}^n$ is a given function.

We let

$$D = \{\psi: [-r,0] \to \mathbb{R}^n, \psi \; is \; continuous \; everywhere \; except \; for \; a \; finite$$
$$number \; of \; points \; \bar{t} \; at \; which \; \psi(\bar{t}) and \; \psi(\bar{t}^+) \; exist, and \; \psi(\bar{t}^-) = \psi(\bar{t})\}$$

$\varphi \in D$; $0 < r < \infty, \tau_k: \mathbb{R}^n \to \mathbb{R}$, $I_k: \mathbb{R}^n \to \mathbb{R}^n, k = 1, \dots, m$ are given functions satisfying some assumptions that will be specified later.

Impulsive differential equations have been studied extensively in recent years. Such equations arise in many applications such as spacecraft control, impact mechanics, chemical engineering and inspection process in operations research. Especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs of Bainov and Simeonov, Lakshmikantham et al, and Samoilenko and Perestyuk, the papers of Benchohra et al and the references therein. The theory of impulsive differential equations with variable time is relatively less developed due to the difficulties created by the state-dependent impulses. Recently, some interesting extensions to impulsive differential equations with variable times have been done by Bajo and Liz, Frigon and O'Regan, Kaul et al, Kaul and Liu, Lakshmikantham et al, Liu and Ballinger and the references cited therein.

Preliminaries

Consider

$$\Omega_{\mathrm{a}} = \{\mathrm{y}: [\mathrm{a} - \mathrm{r}, \mathrm{T}] \to \mathbb{R}^{\mathrm{n}}, \quad a - r < T,$$
$$y(t) \; is \; continuous \; everywhere \; except \; for \; some \; t_k at \; which$$
$$y(t_k^-) and \; y(t_k^+), k = 1, \dots, m \; exist \; and \; y(t_k^-) = y(t_k)\}$$

$$\Omega_a^1 = \{\, y \in \Omega_a,\ \ y\ is\ differentiable\ almost\ everywhere\ on$$
$$[a-r,T), and\ y' \in L^1_{loc}[a-r,T)\}$$
$$AC_{loc}([0,T),\mathbb{R}^n)\ is\ the\ set\ of\ functions\ y \in C([0,T),\mathbb{R}^n$$
$$which\ are\ absolutely\ continuous\ on\ every\ compact\ subset\ of\ [0,T)$$

Throughout this section we will assume that the following conditions hold:

- H1) $f:[0,T)\times D \to \mathbb{R}^n$ is an $L^1_{loc}[0,T) - Carath\acute{e}odory$ function, by this we mean
  - a) The map $t \to f(t,y)$ is measurable for all $y \in D$
  - b) The map $y \to f(t,y)$ is continuous almost all $t \in [0,T)$
  - c) Foe each $r > 0$ there exists $\mu_r \in L^1_{loc}[0,T)$ such that $|y| < r$ implies $|f(t,y)| \le \mu_r(t)$ for almost all $t \in [0,T)$
- H2) the functions $\tau_k \in C^1(\mathbb{R}^n,\mathbb{R})$ for $k = 1,\dots,m$. Moreover
$$0 < \tau_1(x) < \tau_2(x) < \cdots < \tau_m(x) < T\ for\ all\ x \in \mathbb{R}^n$$
- H3) there exist constants c_k such that $|I_k(x)| \le c_k$, $k = 1,\dots,m$, for each $x \in \mathbb{R}^n$.
- H4) $|f(t,y)| \le q(t)\psi(|y|)$

For almost all $t \in [0,T)$ with

$\psi:[0,\infty) \to (0,\infty)$ a Borel measurable function; $\frac{1}{\psi} \in L^1_{loc}[0,\infty)$ and

$q \in L^1_{loc}([0,\infty),\mathbb{R}_+)$ and

$$\int_0^{t^*} q(s)ds < \int_{\varphi(0)}^{\infty} \frac{du}{\psi(u)}\ \ for\ any\ \ t^* < T\ \ and\ \ \varphi(0) = \varphi_0$$

- H5) for all $(t,x) \in [0,T]\times\mathbb{R}^n$ and for all $y_t \in D$
$$\langle \tau'_k(x), f(t,y_t)\rangle \ne 1\ \ \ for\ \ k = 1,\dots,m$$
Where $\langle .,.\rangle$ denotes the scalar product in $\mathbb{R}^n$.
- H6) for all $x \in \mathbb{R}^n$
$$\tau_k\big(I_k(x)\big) \le \tau_k(x) < \tau_{k+1}\big(I_k(x)\big)\ \ \ for\ \ \ k = 1,\dots,m$$

Theorem 1

under the assumptions (H1)-(H6), the problem (1.1) has at leastone solution on [0,T]

Proof. The proof will be given in several steps;

Step 1;

Consider the problem

$$y'(t) = f(t,y_t)\quad a.e\ \ t \in [0,T)$$
$$y(t) = \varphi(t)\ \ \ t \in [-r,0] \qquad\qquad (1.2)$$

which will be needed when we examine the IDE (1.1) we use the Schauder-Tychonoff theorem to establish existence results of (1.2) for completeness we state the fixed point result.

Theorem 2

Let K be a closed convex subset of a locally convex linear topological space E. Assume that $f: K \to K$ is continuous and that $f(K)$ is relatively compact in E. Then f has at least one fixed point in K.

Transform the problem (1.2) into a fixed point problem.

Consider the operator $N: \Omega_0 \to \Omega_0$ defined by

$$N(y)(t) = \begin{cases} \varphi(t) & t \in [-r,0] \\ \varphi(0) + \int_0^t f(s,y_s)ds & t \in [0,T] \end{cases}$$

Let

$$K = \{y \in \Omega_0:\ |y(t)| \le\ b(t),\quad t \in [0,T)\}$$

Where

$$b(t) = J^{-1}\left(\int_0^t q(x)dx\right)$$

$$J(z) = \int_{\varphi_0}^{z} \frac{dx}{\psi(x)}$$

Notice K is a closed, convex, bounded subset of Ω_0.

We next claim that N maps K into K. To see this let $y \in K$. Notice for $t < T$ that

$$|Ny(t)| \leq |\varphi_0| + \int_0^t q(s)\psi(|y(s)|)ds \leq |\varphi_0| + \int_0^t q(s)\psi\big(b(s)\big)ds$$

$$= |\varphi_0| + \int_0^t b'(s)ds = b(t)$$

Thus, $Ny \in K$ and so $N: K \to K$.

It remains to show that $N: \Omega_0 \to \Omega_0$ is continuous and completely continuous.

Claim I: **N** is continuous

Claim II: **N** maps bounded set into bounded set in Ω_0

Claim III: **N** maps bounded sets into equicontinuous sets of Ω_0

As a consequence of Claims I to III together with the Arzela-Ascoli theorem we can conclude that $N: \Omega_0 \to \Omega_0$ is completely continuous.

The Schauder-Tychonoff theorem implies that N has a fixed point in , i.e. (1.2) has a solution $y \in \Omega_0$. denote this solution by y_1.

Define the function

$$r_{k,1}(t) = \tau_k\big(y_1(t)\big) - t, \quad t \in [0, T]$$

(H2) implies that

$$r_{k,1}(0) \neq 0 \ \ for \ \ k = 1, \dots, m$$

If $r_{k,1} \neq 0$ on $[0, T]$, $k = 1, \dots, m$; i,e. $t \neq \tau_k(y_1(t))$ on $[0, T]$ and for $k = 1, \dots, m$; then y_1 is a solution of problem (1.1).

It remains to consider the case when

$$r_{1,1}(t) = 0 \quad for\ some\ \ t \in [0, T]$$

Since $r_{1,1}(0) \neq 0$ and $r_{1,1}$ is continuous, there exists $t_1 > 0$ such that $r_{1,1}(t_1) = 0$ and $r_{1,1}(t) \neq 0$ for all $t \in [0, t_1)$. Thus by (H2) we have $r_{k,1}(t) \neq 0$, for all $t \in [0, t_1)$ and $k = 1, \dots, m$.

Impulsive Functional Differential Equations

In this section various existence results are established for the impulsive functional differential equation

$$y'(t) = f(t, y_t) \qquad a.e\ \ t \in [t_1, T]$$
$$y(t_1^+) = I_1\big(y_1(t_1)\big)(2.1)$$
$$y(t) = y_1(t) \qquad t \in [t_1 - r, t_1]$$

Transform problem (2.1) into a fixed point problem.

Consider the operator $N_1: \Omega_{t_1} \to \Omega_{t_1}$ defined by

$$N_1(y)(t) = \begin{cases} y_1(t) & if \quad t \in [0, t_1] \\ I_1\big(y(t_1)\big) + \int_{t_1}^{t} f(s, y_s)ds & if \ \ t \in (t_1, T] \end{cases}$$

As in section 1, we can show that N_1 is completely continuous. and the set

$$K_1 = \{y \in \Omega_{t_1}, \quad |y(t)| \le b(t), \quad t \in [t_1 - r, T]\}$$

is closed, convex, bounded subset of Ω_{t_1} where

$$b(t) = J^{-1}\left(\int_{t_1}^{t} q(x)dx\right)$$

$$J(z) = \int_{\varphi_0}^{z} \frac{dx}{\psi(x)}$$

Thus $N_1 : K_1 \to K_1$.

As a consequence of the Schauder-Tychonoff theorem, we deduce that N_1 has a fixed point y which is a solution to problem (2.1). Denote this solution by y_2.

Define

$$r_{k,2}(t) = \tau_k\big(y_2(t)\big) - t \quad for\ t \ge t_1$$

If

$$r_{k,2}(t) \neq 0 \quad on \quad (t_1, T], \quad for\ all\ \ k = 1, \dots, m$$

Then

$$y(t) = \begin{cases} y_1(t) & if \quad t \in [0, t_1], \\ y_2(t) & if \quad t \in (t_1, T] \end{cases}$$

is a solution of problem (2.1).

It remains to consider the case when there exists $t > t_1$ with

$$r_{k,2}(t) = 0, \quad k = 1, \dots, m$$

by (H6) we have

$$r_{k,2}(t_1^+) = \tau_k\big(y_2(t_1^+)\big) - t_1 = \tau_k\Big(I_1\big(y_1(t_1)\big)\Big) - t_1$$
$$> \tau_{k-1}\big(y_1(t_1)\big) - t_1$$
$$\ge \tau_1\big(y_1(t_1)\big) - t_1 = r_{1,1}(t_1) = 0$$

Since $r_{k,2}$ is continuous, there exists $t_2 > t_1$ such that

$$r_{k,2}(t_2) = 0, \quad r_{k,2}(t) \neq 0 \ \ for\ all\ \ t \in (t_1, t_2)$$

Suppose now that there is $t^* \in (t_1, t_2)$ such that

$$r_{1,2}(t^*) = 0$$

from (H6), it follows that

$$r_{1,2}(t_1^+) = \tau_1\big(y_2(t_1^+)\big) - t_1 = \tau_1\Big(I_1\big(y_1(t_1)\big)\Big) - t_1$$
$$\le \tau_1\big(y_1(t_1)\big) - t_1 = r_{1,1}(t_1) = 0$$

Thus the function $r_{1,2}$ attains a nonnegative maximum at some point $t_1^* \in (t_1, T]$.

Since

$$y'_2(t) = f(t, y_2(t))$$

Then

$$r'_{1,2}(t_1^*) = \tau'_1\big(y_2(t_1^*)\big)y'_2(t_1^*) - 1 = \tau'_1\big(y_2(t_1^*)\big)f\big(t_1^*, y_2(t_1^*)\big) - 1 = 0$$

Therefore

$$\langle \tau'_1\big(y_2(t_1^*)\big), f(t_1^*, y_2(t_1^*))\rangle = 1$$

which is a contradiction by (H5).

Continue this process and the result of the theorem follows. Observe that if $T < \infty$ the process will stop after a finite number of steps taking into account that $y_{m+1} \coloneqq y\big|_{[t_m, T]}$ is a solution to the problem

$$\begin{aligned} &y'(t) = f(t, y_t) \qquad a.e \ \ t \in (t_m, T) \\ &y(t_m^+) = I_m\big(y_{m-1}(t_m)\big) \qquad\qquad (3.1) \\ &y(t) = y_{m-1}(t) \qquad t \in [t_m - r, t_m] \end{aligned}$$

The solution y of the problem (1.1) is then defined by

$$y(t) = \begin{cases} y_1(t) & if \quad t \in [-r, t_1] \\ y_2(t) & if \quad t \in (t_1, t_2] \\ \vdots & \\ y_{m+1}(t) & if \quad t \in (t_m, T] \end{cases}$$

References:

Bainov D. D. & Dishliev A. B., Sufficient condition for absence of beating in systems of differential equations with impulse, *Appl. Anal*. **18** (1984), 67-73.

Bainov D. D. & Dishliev A. B., Conditions for the absence of phenomenon beating for systems of impulse differential equations, *bull. Inst. Math. Acad. Sinica*. **13**(1985), 237-256.

Bainov D. D. & Simeonov P. S., *Systems with Impulse Effect, Stability Theory and Applications*. Ellis Horwood, Chichester (1989).

Benchohra M., Henderson J., Ntouyas S. & Ouahab A., Impulsive Functional Differential Equations with Variable Times, *Computers & Mathematics with Applications*. **47**(2004), no. 10-11, 1659-1665.

Benchohra M., Henderson J., Ntouyas S. & Ouahab A., Impulsive functional dynamic equations on time scales with infinity delay, *Dynam. Systems and Appl*. **14**(2005), no. 1-10.

Franco D., Liz E. & Nieto J. J. A contribution to the study of functional differential equations
with impulses, *Math. Nachr*. **28**(2000), 49-60.

Frigon M. & O'Regan D., Impulsive Differential Equations with Variable Times. *Nonlinear Analysis: Theory, Methods & Applications*, **26**(12):1913-1922, 1996.

Granas and J. Dugundji, *Fixed Point Theory*. SpringerMonographs inMathematics, Springer, New York, 2003.

Lakshmikantham V., Bainov D. D. & Simeonov P. S., *Theory of Impulsive Differential Equations*. World Scientific Press, Singapore (1989).

Samoilenko A. M. & Perestyuk, *Impulsive Differential Equations*. World Scientific, Singapore (1995).

Vatsala A. S. & Sun Y., Periodic boundary value problem of impulsive differential equations, *Applacable Analysis*. **44**(1992), no.3-4, 145-158.