A RECENT APPROACH TO CONTINUOUS TIME OPEN LOOP STACKELBERG DYNAMIC GAME WITH MIN-MAX COOPERATIVE AND NONCOOPERATIVE FOLLOWERS

M.S.Osman, Prof.
10^{th} Of Ramadan University, Egypt
N.A. El-Kholy, Dr.
Tanta University, Faculty Of Science, Egypt
E.I.Soliman, PhD Student
Aastmt, College Of Engineering, Alexandria, Egypt

Abstract

Stackelberg games play an extremely important role in such fields as economics, management, politics and behavioral sciences. There exists extensive literature about static optimization problems. However, the studies on dynamic optimization problems are relatively scarce in spite of the importance in explaining and predicting some phenomena rationally. In this paper, new approach of continuous time open loop stackelberg differential game introduced with a min-max game (secure concept) between multiple followers playing dependently (the pareto followers) and others independently (the nash followers), the problem formulation is presented with an example for the solution of the game.

Keywords: Dynamic game; stackelberg differential game; pareto, nash

Introduction

Stackelberg (leader–follower) games have a wide variety of applications. In a stackelberg game, one player acts as a leader and the rest as followers. The problem is then to find an optimal strategy for the leader, assuming that the Followers react in such a rational way that they optimize their objective functions given the leader's actions. This is the static optimization model introduced by von stackelberg (Von Stackelberg, 1952). When players interact by playing a similar stage game numerous times, the game is called a dynamic, or repeated game. Unlike static games, players have at least some information about the strategies chosen on others and thus may contingent their play on past moves. Dynamic optimization was first considered by (Chen, Cruz, 1972) and (Simaan, Cruz, 1973a), and subsequently studied by a number of authors (Basar, Olsder, 1995), (Li, Cruz, Simaan, 2002), (Nie, Chen, Fukushima, 2006), (Nie, 2005), (Fent, Feichtinger, Tragler, 2002) and (Feichtinger, Grienauer, Tragler, 2002). In (Chen and Cruz, 1972) and (Simaan, Cruz, 1973a), necessary and sufficient conditions for the stackelberg games with open-loop information have been obtained and explicit solutions are given. In (Simaan, Cruz, 1973b), stackelberg solution is extended to multi-players and necessary conditions for the existence of an open-loop stackelberg games are shown. In this paper we represent new approach for such game model with an illustrative example.

Problem formulation

Consider open - loop stakelberg differential game with $M+N$ followers. Under the concept of coalition we suppose that the M of these players agree to form a coalition and cooperate as a single player to minimize their collective costs playing a min - max differential game against the others N nash followers outside the coalition.let

$u(t) \in R^{l}$ be the composite control of the players inside the coalition, $v(t) \in R^{m}$ be the composite control of the players outside the coalition, and $u_o \in R^{s-(l+m)}$ be the strategy of the leader where the composite control $(u_o, u, v) \in R^{s}$ is an element of the constrain set $\Omega = \{(u_o, u, v) \in R^{s} \mid \dot{x}(t) = f(t, x(t), u_o(t), u(t), v(t)), x(t_0) = x_0, h(t, x, u_o, u, v) \geq 0\}$, where

$$\dot{x}(t) = f(t, x(t), u_o(t), u(t), v(t))$$

Represent the system of nonlinear differential equations that govern the game motion, and

$$J_0(t, x, u_o, u, v) = \phi_0(x(t_f)) + \int_{t_0}^{t_f} I_0(t, x(t), u_o(t), u(t), v(t))$$

$$\overline{J}_i(t, x, u_o, u, v) = \overline{\phi}_i(x(t_f)) + \int_{t_0}^{t_f} \overline{I}_i(t, x(t), u_o(t), u(t), v(t)) \quad , i = 1, 2, \cdots, M$$

$$J_j(t, x, u_o, u, v) = \phi_j(x(t_f)) + \int_{t_0}^{t_f} I_j(t, x(t), u_o(t), u(t), v(t)) \quad , j = 1, 2, \cdots, N$$

are the cost functions for the leader and the $M+N$ followers respectively, where $[t_0, t_f]$ denotes the fixed prescribed duration of the game,

$x(t) \in R^{n}$ is the state vector of the game,

$h(.) : [t_0, t_f] \times R^{n} \times R^{s} \rightarrow R^{q}$

$f(.) : [t_0, t_f] \times R^{n} \times R^{s} \rightarrow R^{n}$

$I_j(.):[t_0,t_f]\times R^n \times R^s \to R,\ j=1,2,\cdots,N.$

$\overline{I}_i(.):[t_0,t_f]\times R^n \times R^s \to R,\ i=1,2,\cdots,M.$

$\phi_j(.):R^n \to R,\ j=1,2,\cdots,N.$

$\overline{\phi}_i(.):R^n \to R,\ i=1,2,\cdots,M.$

Are continuous differentiable functions.

Definition 1.1 *the pareto response set* P *for the* M *followers, is the set of all* $(u_o,\hat{u},v)\in\Omega\subseteq R^s$ *such that for each* $i=1,2,\cdots,M$. *there does not exist* $(u_o,u,v)\in\Omega$ *such that*

$$\overline{J}_i(u_o,\hat{u},v) = \overline{J}_i(u_o,u,v)$$

And for at least one *i*

$$\overline{J}_i(u_o,\hat{u},v) \prec \overline{J}_i(u_o,u,v)$$

Definition 1.2 *the nash response set* Đ *for the* N *followers, is the set of all* $(u_o,u,\hat{v})\in\Omega\subseteq R^s$ *such that for each* $j=1,2,\cdots,N$

$$J_j(u_o,u,\hat{v}) \le J_j(u_o,u,v_j,\hat{v}_{-j})$$

for all $(v_j,\hat{v}_{-j})\in R^m$ such that $(u_o,u,v_j,\hat{v}_{-j})\in\Omega$ where $\hat{v}_{-j}=(\hat{v}_1,\hat{v}_2,\cdots,\hat{v}_{j-1},\hat{v}_{j+1},\cdots,\hat{v}_N)$

Definition 1.3 *a control* $(u_o,u^*,\overline{v})\in\Omega\subseteq R^s$ *is a min-max solution for the player* i *of the m followers if and only if* $(u_o,u^*,\overline{v})\in P$ *and*

$$\overline{J}_i(u_o,u^*,v) \le \overline{J}_i(u_o,u^*,\overline{v})$$

for all $(u_o,u^*,v)\in\Omega$

Definition 1.4 *a control* $(u_o,\overline{u}^j,v^*)\in\Omega\subseteq R^s$ *is a min-max solution for the player* j *of the n followers if and only if* $(u_o,\overline{u}^j,v^*)\in$ Đ *and*

$$J_j(u_o,u,v^*) \le J_j(u_o,\overline{u}^j,v^*)$$

for all $(u_o,u,v^*)\in\Omega$

Definition 1.5 *a control* $(u_o^*,u^*,v^*)\in\Omega\subseteq R^s$ *is a stackelberg solution for the leader of the defined game if and only if* $(u_o^*,u^*,v^*)\in$ Đ $\cap P$ *and*

$$J_o(u_o^*,u^*,v^*) \le J_o(u_o,u,v) \quad \text{for all } (u_o,u,v)\in \text{Đ}\cap P.$$

the problem is formally stated as follows:

1. Find $(u_o^*,\hat{u},v^*)$ and $(u_o^*,u^*,\overline{v})$ that solves the problems:

$$P_1:\min_u \overline{J}(t,u_o^*,u,v^*) = \sum_{i=1}^{M}\overline{w}_i\overline{J}_i(t,u_o^*,u,v^*)$$

$$= \overline{\phi}(x(t_f)) + \int_{t_0}^{t_f} \overline{I}(t, x, u_o^*, u, v^*)\, dt$$

subject to

$$\Omega = \{(u_o^*, u, v^*) \in R^s \mid \dot{x}(t) = f(t, x, u_o^*, u, v^*), x(t_0) = x_0, h(t, x, u_o^*, u, v^*) \geq 0\}$$

$$P_2 : \min_u \max_v \overline{J}(t, u_o^*, u, v) = \sum_{i=1}^{M} \overline{w}_i \overline{J}_i(t, u_o^*, u, v)$$

$$= \overline{\phi}(x(t_f)) + \int_{t_0}^{t_f} \overline{I}(t, x, u_o^*, u, v)\, dt$$

subject to

$$\Omega = \{(u_o^*, u, v) \in R^s \mid \dot{x}(t) = f(t, x, u_o^*, u, v), x(t_0) = x_0, h(t, x, u_o^*, u, v) \geq 0\}$$

respectively, where

$$\overline{J}_i(u_o(t), u(t), v(t)) = \overline{\phi}_i(x(t_f)) + \int_{t_0}^{t_f} \overline{I}_i[t, x(t), u_o(t), u(t), v(t)]dt, i = 1, 2, \cdots, M$$

are the cost functions for the M players, $\overline{\phi}(x(t_f)) = \sum_{i=1}^{M} \overline{w}_i \overline{\phi}_i(x(t_f))$ and $\overline{I}(t, x, u_o^*, u, v^*) = \sum_{i=1}^{M} \overline{w}_i \overline{I}_i(t, x, u_o^*, u, v^*)$ for each $\overline{w} \in \overline{W}$, where $\overline{W} = \{\overline{w} \in R^M \mid \overline{w}_i \geq 0, \sum_{i=1}^{M} \overline{w}_i = 1\}$,

2. Find $(u_o^*, u^*, \hat{v})$ and $(u_o^*, \overline{u}^j, v^*)$ that solves the problems:

$$P_3 : \min_{v_j} J_j(t, u_o^*, u^*, v_j, \hat{v}_{-j}) = \phi_j(x(t_f)) + \int_{t_0}^{t_f} I_j(t, x, u_o^*, u^*, v_j, \hat{v}_{-j})\, dt$$

subject to

$$\Omega = \{(u_o^*, u^*, v_j, \hat{v}_{-j}) \in R^s \mid \dot{x}(t) = f(t, x, u_o^*, u^*, v_j, \hat{v}_{-j}), x(t_0) = x_0, h(t, x, u_o^*, u^*, v_j, \hat{v}_{-j}) \geq 0\}$$

$$P_4 : \min_{v_j} \max_u J_j(t, u_o^*, u, v_j, v_{-j}^*) = \phi_j(x(t_f)) + \int_{t_0}^{t_f} I_j(t, x, u_o^*, u, v_j, v_{-j}^*)\, dt$$

subject to

$$\Omega = \{(u_o^*, u, v_j, v_{-j}^*) \in R^s \mid \dot{x}(t) = f(t, x, u_o^*, u, v_j, v_{-j}^*), x(t_0) = x_0, h(t, x, u_o^*, u, v_j, v_{-j}^*) \geq 0\}$$

respectively,

3. Find (u_o^*, u^*, v^*) that solves the problems P_1, P_3 and

$$P_5 : \min_{u_o} J_0(t, x, u_o, u^*, v^*) = \phi_0(x(t_f)) + \int_{t_0}^{t_f} I_0(t, x(t), u_o(t), u(t)^*, v(t)^*)$$

subject to

$$\Omega = \{(u_o, u^*, v^*) \in R^s \mid \dot{x}(t) = f(t, x, u_o, u^*, v^*), x(t_0) = x_0, h(t, x, u_o, u^*, v^*) \geq 0\}$$

where, $x(t):[t_0,t_f]\rightarrow R^n$ the state of the system is assumed to be a piecewise continuous function of t, and the functions
$f:[t_0,t_f]\times R^n\times R^s\rightarrow R^n$, ϕ_0, ϕ_j,
and $\overline{\phi_i}$ are continuous functions on R^n.

Theorem 1

if $(u_o^*,\hat{u},v)\in\Omega\subseteq R^s$ is pareto solution for the m followers inside the coalition with state trajectory x^* corresponding to problem P_1, then there exist continuous costate functions $p:[t_0,t_f]\rightarrow R^n$, $\overline{\delta}\in R^q$ such that the following relations are satisfied

$$\dot{x}^*(t)=f(t,x^*,u_o^*,\hat{u},v)\quad ,x^*(t_0)=x_0$$

$$\dot{p}(t)=-\frac{\partial\overline{H}(t,x^*,u_o^*,\hat{u},v,\overline{w}_1,\cdots,\overline{w}_M,p,\overline{\delta})}{\partial x(t)}$$

$$p(t_f)=\frac{\partial\overline{\phi}(x^*(t_f))}{\partial x(t_f)},\quad \overline{\phi}=\sum_{i=1}^{M}\overline{w}_i\overline{\phi}_i(x^*(t_f))$$

$$\frac{\partial\overline{H}(t,x^*,u_o^*,\hat{u},v,\overline{w}_1,\cdots,\overline{w}_M,p,\overline{\delta})}{\partial u}=0$$

$$\overline{\delta}h(t,x^*,u_o^*,\hat{u},v)=0$$

$$h(t,x^*,u_o^*,\hat{u},v)\geq 0$$

$$\overline{\delta}\geq 0$$

where

$$\overline{H}(t,x,u_o,u,v,\overline{w}_1,\cdots,\overline{w}_M,p,\overline{\delta})=\sum_{i=1}^{M}\overline{w}_i\overline{I}_i(t,x,u_o,u,v)+p^T f(t,x,u,v)$$
$$-\overline{\delta}^T h(t,x,u_o,u,v)$$

is the hamiltonian function of the followers inside the coalition and its partial derivatives.

Furthermore, if $(u_o^*,u^*,\overline{v})\in\Omega\subseteq R^s$ is min-max point for the pareto followers with state trajectory x^* corresponding to problems P_2, then there exist continuous costate functions $p:[t_0,t_f]\rightarrow R^n$, $\overline{\varsigma}\in R^q$ and $\overline{\eta}^j\in R^q$ such that the following relations are satisfied

$$\dot{x}^*(t)=f(t,x^*,u_o^*,u^*,\overline{v}),x^*(t_0)=x_0$$

$$\dot{p}(t)=-\frac{\partial\overline{H}(t,x^*,u_o^*,u^*,\overline{v},\overline{w}_1,\cdots,\overline{w}_M,p,\overline{\varsigma},\overline{\eta}^j)}{\partial x(t)}$$

$$p(t_f) = \frac{\partial \bar{\phi}(x^*(t_f))}{\partial x(t_f)}, \quad \bar{\phi} = \sum_{i=1}^{M} \bar{w}_i \bar{\phi}_i(x^*(t_f))$$

$$\frac{\partial \bar{H}(t, x^*, u_o^*, u^*, \bar{v}, \bar{w}_1, \cdots, \bar{w}_M, p, \bar{\varsigma}, \bar{\eta}^j)}{\partial u} = 0$$

$$\frac{\partial \bar{H}(t, x^*, u_o^*, u^*, \bar{v}, \bar{w}_1, \cdots, \bar{w}_M, p, \bar{\varsigma}, \bar{\eta}^j)}{\partial v_j} = 0, \quad j = 1, 2, \cdots, N.$$

$$\bar{\varsigma} h(t, x^*, u_o^*, u^*, \bar{v}) = 0$$

$$\bar{\eta}^j h(t, x^*, u_o^*, u^*, \bar{v}) = 0, \ j = 1, 2, \cdots, N.$$

$$h(t, x^*, u_o^*, u^*, \bar{v}) \geq 0$$

$$\bar{\varsigma} \geq 0$$

$$\bar{\eta}^j \leq 0, \ j = 1, 2, \cdots, N.$$

where

$$\bar{H}(t, x, u_o, u, v, \bar{w}_1, \cdots, \bar{w}_M, p, \bar{\varsigma}, \bar{\eta}) = \sum_{i=1}^{M} \bar{w}_i \bar{I}_i(t, x, u_o, u, v) + p^T f(t, x, u_o, u, v) - \bar{\varsigma}^T h(t, x, u_o, u, v)$$

is the hamiltonian function of the followers inside the coalition and its partial derivatives evaluated using the two sets of multipliers $\bar{\varsigma}$ and $\bar{\eta}^j$.

Now for the followers outside the coalition, if $(u_o^*, u, \hat{v}) \in \Omega \subseteq R^s$ is nash equilibrium point for the N players corresponding to problem P_3, then there exist continuous costate functions $q^j : [t_0, t_f] \to R^n$, $\delta^j \in R^q$ such that the following relations are satisfied

$$\dot{x}^*(t) = f(t, x^*, u_o^*, u, \hat{v}), x^*(t_0) = x_0$$

$$\dot{q}_k^j(t) = -\frac{\partial H_j(t, x^*, u_o^*, u, \hat{v}, q^j, \delta^j)}{\partial x_k(t)}$$

$$q_k^j(t_f) = \frac{\partial \phi_j(x^*(t_f))}{\partial x_k(t_f)}, \qquad k = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, N$$

$$\frac{\partial H_j(t, x^*, u_o^*, u, \hat{v}, q^j, \delta^j)}{\partial v_j} = 0, \qquad j = 1, 2, \cdots, N.$$

$$\delta^j h(t, x^*, u_o^*, u, \hat{v}) = 0, \ j = 1, 2, \cdots, N$$

$$h(t, x^*, u_o^*, u, \hat{v}) \geq 0$$

$$\delta^j \geq 0, \ j = 1, 2, \cdots, N$$

where

$$H_j(t,x,u_o,u,v,q^j,\delta^j) = I_j(t,x,u_o,u,v) + q^{jT} f(t,x,u_o,u,v)$$
$$-\delta^{jT} h(t,x,u_o,u,v), \quad j = 1,2,\cdots,N.$$

are the hamiltonian functions of each follower and its partial derivatives.

Furthermore, if $(u_o^*,\bar{u}^j,v^*) \in \Omega \subseteq R^s$ is a min-max point for player j of the nash players corresponding to problem P_4, then there exist continuous costate functions $q^j:[t_0,t_f] \to R^n$, $\varsigma^j \in R^q$ and $\eta^j \in R^q$ such that the following relations are satisfied

$$\dot{x}^*(t) = f(t,x^*,u_o^*,\bar{u}^j,v^*), x^*(t_0) = x_0$$
$$\dot{q}_k^j(t) = -\frac{\partial H_j(t,x^*,u_o^*,\bar{u}^j,v^*,q^j,\varsigma^j,\eta^j)}{\partial x_k(t)}$$
$$q_k^j(t_f) = \frac{\partial \phi_j(x^*(t_f))}{\partial x_k(t_f)}, k = 1,2,\cdots,n, \quad j = 1,2,\cdots,N$$
$$\frac{\partial H_{\hat{j}}(t,x^*,u_o^*,\bar{u}^j,v^*,q^j,\varsigma^j,\eta^j)}{\partial v_{\hat{j}}} = 0, j = 1,2,\cdots,N, \quad \hat{j} = 1,2,\cdots,N$$
$$\frac{\partial H_j(t,x^*,u_o^*,\bar{u}^j,v^*,q^j,\varsigma^j,\eta^j)}{\partial u} = 0, j = 1,2,\cdots,N$$
$$\varsigma^{\hat{j}} h(t,x^*,u_o^*,\bar{u}^j,v^*) = 0, j = 1,2,\cdots,N, \quad \hat{j} = 1,2,\cdots,N$$
$$\eta^j h(t,x^*,u_o^*,\bar{u}^j,v^*) = 0, j = 1,2,\cdots,N$$
$$h(t,x^*,u_o^*,\bar{u}^j,v^*) \geq 0$$
$$\varsigma^j \geq 0, j = 1,2,\cdots,N$$
$$\eta^j \leq 0, j = 1,2,\cdots,N.$$

where

$$H_j(t,x,u_o,u,v,q^j,\varsigma^j,\eta^j) = I_j(t,x,u_o,u,v) + q^{jT} f(t,x,u_o,u,v)$$
$$-\varsigma^{jT} h(t,x,u_o,u,v), \quad j = 1,2,\cdots,N.$$

are the hamiltonian functions of the nash players and its partial derivatives evaluated using the two sets of multipliers ς^j and η^j.

Theorem 2

if $(u_o^*,u^*,\bar{v}) \in \Omega \subseteq R^s$ and $(u_o^*,\bar{u}^j,v^*) \in \Omega \subseteq R^s$ are the min-max points corresponding to problems P_2 and P_4 respectively, then $(u_o^*,u^*,v^*) \in \Omega \subseteq R^s$ is an interior open loop stackelberg solution for the leader with pareto and nash followers, and x^* the corresponding state trajectory, if there exist continuous costate functions $p:[t_0,t_f] \to R^n$ and $q^j:[t_0,t_f] \to R^n$ such that

$$\dot{x}^*(t) = f(t,x^*,u_o^*,u^*,v^*), x^*(t_0) = x_0$$

$$\dot{p}(t) = -\frac{\partial \overline{H}(t, x^*, u_o^*, u^*, v^*, \overline{w}_1, \cdots, \overline{w}_M, p, \hat{\delta})}{\partial x(t)}$$

$$p(t_f) = \frac{\partial \overline{\phi}(x^*(t_f))}{\partial x(t_f)}, \quad \overline{\phi} = \sum_{i=1}^{M} \overline{w}_i \overline{\phi}_i (x^*(t_f))$$

$$\frac{\partial \overline{H}(t, x^*, u_o^*, u^*, v^*, \overline{w}_1, \cdots, \overline{w}_M, p, \hat{\delta})}{\partial u} = 0$$

$$\dot{q}_k^j(t) = -\frac{\partial H_j(t, x^*, u_o^*, u^*, v^*, q^j, \hat{\delta}^j)}{\partial x_k(t)}, k = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, N.$$

$$q_k^j(t_f) = \frac{\partial \phi_j(x^*(t_f))}{\partial x_k(t_f)}, k = 1, 2, \cdots, n \quad, j = 1, 2, \cdots, N.$$

$$\frac{\partial H_j(t, x^*, u_o^*, u^*, v^*, q^j, \hat{\delta}^j)}{\partial v_j} = 0, j = 1, 2, \cdots, N.$$

$$h(t, x^*, u_o^*, u^*, v^*) \geq 0$$

$$\hat{\delta} h(t, x^*, u_o^*, u^*, v^*) = 0$$

$$\hat{\delta}^j h(t, x^*, u_o^*, u^*, v^*) = 0, j = 1, 2, \cdots, N.$$

$$\hat{\delta} \geq 0$$

$$\hat{\delta}^j \geq 0, j = 1, 2, \cdots, N.$$

where

$$\overline{H}(t, x, u_o, u, v, \overline{w}_1, \cdots, \overline{w}_M, p, \hat{\delta}) = \sum_{j=1}^{M} \overline{w}_j \overline{I}_j(t, x, u_o, u, v) + p^T f(t, x, u_o, u, v) - \hat{\delta} h(t, x, u_o, u, v)$$

and

$$H_j(t, x, u_o, u, v, q^j, \hat{\delta}^j) = I_j(t, x, u_o, u, v) + q^{jT} f(t, x, u_o, u, v) - \hat{\delta}^j h(t, x, u_o, u, v), \quad j = 1, 2, \cdots, N.$$

are the hamiltonian functions of the followers inside and outside the coalition respectively.

Furthermore, for the leader, there exists continuously differentiable functions (costate vectors), $p_0(t):[t_0, t_f] \to R^n$, $\rho(t):[t_0, t_f] \to R^n$, $\sigma_j(t):[t_0, t_f] \to R^n$ and continuous functions $\overline{\lambda}(t):[t_0, t_f] \to R^l$, $\lambda_j(t):[t_0, t_f] \to R^{m_j}, m = \sum_{j=1}^{N} m_j$ (lagrange multipliers), such that the following are satisfied

$$\dot{x}^*(t) = f(t, x^*, u_o^*, u^*, v^*) \quad, x^*(t_0) = x_0$$

$$\dot{p}_0(t) = -\frac{\partial H_0(t, x^*, u_o^*, u^*, v^*, p_0, p, \rho, \overline{\lambda}, \lambda_j, \sigma_j, q^j)}{\partial x}$$

$$\dot{\rho}(t) = -\frac{\partial H_0(t, x^*, u_o^*, u^*, v^*, p_0, p, \rho, \overline{\lambda}, \lambda_j, \sigma_j, q^j)}{\partial p}$$

$$\dot{\sigma}_j(t) = -\frac{\partial H_0(t, x^*, u_o^*, u^*, v^*, p_0, p, \rho, \overline{\lambda}, \lambda_j, \sigma_j, q^j)}{\partial q^j}$$

$$\frac{\partial H_0(t, x^*, u_o^*, u^*, v^*, p, p_0, p, \rho, \overline{\lambda}, \lambda_j, \sigma_j, q_j)}{\partial u} = 0$$

$$\frac{\partial H_0(t, x^*, u_o^*, u^*, v^*, p, p_0, p, \rho, \overline{\lambda}, \lambda_j, \sigma_j, q^j)}{\partial v_j} = 0$$

$$\frac{\partial H_0(t, x^*, u_o^*, u^*, v^*, p_0, p, \rho, \overline{\lambda}, \lambda_j, \sigma_j, q^j)}{\partial u_o} = 0$$

$$h(t, x^*, u_o^*, u^*, v^*) \geq 0$$

$$\mu h(t, x^*, u_o^*, u^*, v^*) = 0$$

$$\mu \geq 0$$

With the boundry condtions

$$p(t_f) = \frac{\partial \overline{\phi}(x(t_f))}{\partial x(t_f)} \quad , q^j(t_f) = \frac{\partial \phi_j(x(t_f))}{\partial x(t_f)} \quad , p_0(t_f) = \frac{\partial \phi_0(x(t_f))}{\partial x(t_f)}$$

$$\rho(t_0) = 0, \quad \sigma(t_0) = 0$$

where the hamiltonian function of the leader is given by

$$H_0(t, p_0, x, u_o, u, v) = I_0(t, x, u_o, u, v) + p_0^T f(t, x, u_o, u, v) + \rho^T[-\nabla_x \overline{H}] + \overline{\lambda}^T[\nabla_u \overline{H}]$$
$$+ \sum_{j=1}^{N} \left\{ \sigma_j^T[-\nabla_x H_j] + \lambda_j^T[\nabla_{v_j} H_j] \right\} - \mu^T h(t, x, u_o, u, v)$$

Example 2.1 consider the following stakelberg open loop differential game; where the leader announce his control $u_o \in R$; then four followers announce their controls $(u, v) \in R^4$ such that two of the followers with control $u = (u_1, u_2) \in R^2$ decide to cooperate to minimize their collective cost playing a min-max game with each of the last two players outside the coalition with controls $v = (v_1, v_2) \in R^2$ that acting independently to each other, the cost functions are respectively given by :

$$J_o = \int_{t_0}^{t_f} [(u_o - 1)^2 + u_1 - u_2 - v_1 + v_2] dt$$

$$\overline{J}_1 = \int_{t_0}^{t_f} [(u_1 + u_o - 3)^2 + (u_2 - 1)^2 + (v_2 - 3)^2] dt$$

$$\bar{J}_2 = \int_{t_0}^{t_f} [(u_1 - 1)^2 + \frac{1}{2}(u_2 - u_o - 1)^2 + (v_1 - 2)^2]dt$$

$$J_1 = x_1(t_f) + \int_{t_0}^{t_f} [-v_1 u_1 + \frac{1}{2}v_1^2 + u_1]dt$$

and

$$J_2 = \frac{2}{3}x_2(t_f) + \int_{t_0}^{t_f} [(v_2 - u_2)^2 + u_2 - 1]dt$$

S.t.

$$\Omega = \left\{ (u_o, u, v) \in R^5 \mid \quad \dot{x}_1 = \frac{1}{2}v_1, \quad \dot{x}_2 = u_o + v_2, \quad u_i \geq 0, \quad i = 1, 2., \right.$$

$$\left. u_1 + u_2 \leq 4, \quad 0 \leq v_1 \leq 1 \quad and \quad 0 \leq v_2 \leq \frac{2}{3} \right\}$$

According to the cooperative between the first two followers, the hamiltonian function of their collective cost of $\bar{J}_1$ and $\bar{J}_2$ is given by (8)

$$\begin{aligned}
\bar{H}(t, x, u_0, u, v, \bar{w}, p, \bar{\delta}) &= \bar{I} + p^T f(t, x, u_0, u, v) - \bar{\delta}^T h(t, x, u_0, u, v) \\
&= \bar{w}[(u_1 + u_0 - 3)^2 + (u_2 - 1)^2 + (v_2 - 3)^2] \\
&+ (1 - \bar{w})[(u_1 - 1)^2 + \frac{1}{2}(u_2 - u_0 - 1)^2 + (v_1 - 2)^2] + p_1(\frac{1}{2}v_1) \\
&+ p_2(u_0 + v_2) - \bar{\delta}_0 u_0 - \bar{\delta}_1 u_1 - \bar{\delta}_2 u_2 - \bar{\delta}_3(4 - u_1 - u_2) \\
&- \bar{\delta}_4 v_1 - \bar{\delta}_5 v_2 - \bar{\delta}_6(1 - v_1) - \bar{\delta}_7(\frac{2}{3} - v_2)
\end{aligned}$$

where $w_1 = \bar{w}$ and $w_2 = 1 - \bar{w}$, $0 \leq \bar{w} \leq 1$.

Applying necessary conditions for pareto solutions equations (1)-(7)we get

$$\dot{p}_1(t) = -\frac{\partial \bar{H}}{\partial x_1(t)} = 0$$

$$p_1(t_f) = \frac{\partial \bar{\phi}(x(t_f))}{\partial x_1(t_f)} = 0$$

$$\dot{p}_2(t) = -\frac{\partial \bar{H}}{\partial x_2(t)} = 0$$

$$p_2(t_f) = \frac{\partial \bar{\phi}(x(t_f))}{\partial x_2(t_f)} = 0$$

$$\frac{\partial \bar{H}}{\partial u_1} = 2\hat{u}_1 + 2\bar{w}u_0 - 4\bar{w} - 2 - \bar{\delta}_1 + \bar{\delta}_3 = 0$$

$$\frac{\partial \bar{H}}{\partial u_2} = (1 + \bar{w})\hat{u}_2 - (1 - \bar{w})u_0 - (1 + \bar{w}) - \bar{\delta}_2 + \bar{\delta}_3 = 0$$

$$\bar{\delta}_i h_i(t, x^*, u_o, \hat{u}, v) = 0, i = 0, 2, \cdots, 7$$
$$\bar{\delta}_i \geq 0, i = 0, 2, \cdots, 7 \qquad (67)$$

where $h = [u_0, u_1, u_2, 4 - u_1 - u_2, v_1, v_2, 1 - v_1, \frac{2}{3} - v_2]$. Solving the system of equations (67), it yields that $p_1 = 0$, $p_2 = 0$ and the pareto response set p corresponding to $\bar{\delta}_i = 0$ is given by

$$P = \left\{ (u_0, \hat{u}, v) \in \Omega \,|\, \hat{u} = (\hat{u}_1, \hat{u}_2) = \left(\bar{w}(2 - u_0) + 1, \frac{1 - \bar{w}}{1 + \bar{w}} u_0 + 1 \right) \middle| u_0 \in \left[0, \frac{2(1 - \bar{w}^2)}{1 - 2\bar{w} - \bar{w}^2} \right), \right.$$
$$\left. \bar{w} \in \left[0, \frac{\sqrt{6} - 1}{5} \right] \quad or \quad u_0 \in \left[0, \frac{1 + 2\bar{w})}{\bar{w}} \right), \bar{w} \in \left[\frac{\sqrt{6} - 1}{5}, -1 + \sqrt{2} \right) \right\}$$
(68)

for the followers outside the coalition that act independently, (27) gives their hamiltonian functions as follows:

$$H_1(t, x, u_0, u, v, q^j, \delta^1) = I_1 + q^{1T} f - \delta^{1T} h(t, x, u_o, u, v)$$
$$= -v_1 u_1 + \frac{1}{2} v_1^2 + u_1 + q_1^1 (\frac{1}{2} v_1) + q_2^1 (u_0 + v_2)$$
$$-\delta_0^1 u_0 - \delta_1^1 u_1 - \delta_2^1 u_2 - \delta_3^1 (4 - u_1 - u_2) - \delta_4^1 v_1 - \delta_5^1 v_2$$
$$-\delta_6^1 (1 - v_1) - \delta_7^1 (\frac{2}{3} - v_2)$$
$$H_2(t, x, u_0, u, v, q^j, \delta^2) = I_2 + q^{2T} f - \delta^{2T} h(t, x, u_o, u, v)$$
$$= (v_2 - u_2)^2 + u_2 - 1 + q_1^2 (\frac{1}{2} v_1) + q_2^2 (u_0 + v_2)$$
$$-\delta_0^2 u_0 - \delta_1^2 u_1 - \delta_2^2 u_2 - \delta_3^2 (4 - u_1 - u_2) - \delta_4^2 v_1 - \delta_5^2 v_2$$
$$-\delta_6^2 (1 - v_1) - \delta_7^2 (\frac{2}{3} - v_2)$$

and by applying the necessary conditions (20)-(26),

$$\dot{q}_1^1(t) = -\frac{\partial H_1}{\partial x_1(t)} = 0$$
$$q_1^1(t_f) = \frac{\partial \phi_1(x^*(t_f))}{\partial x_1(t_f)} = 1$$
$$\dot{q}_2^1(t) = -\frac{\partial H_1}{\partial x_2(t)} = 0$$
$$q_2^1(t_f) = \frac{\partial \phi_1(x^*(t_f))}{\partial x_2(t_f)} = 0$$

$$\dot{q}_1^2(t) = -\frac{\partial H_2}{\partial x_1(t)} = 0$$

$$q_1^2(t_f) = \frac{\partial \phi_2(x^*(t_f))}{\partial x_1(t_f)} = 0$$

$$\dot{q}_2^2(t) = -\frac{\partial H_2}{\partial x_2(t)} = 0$$

$$q_2^2(t_f) = \frac{\partial \phi_2(x^*(t_f))}{\partial x_2(t_f)} = \frac{2}{3}$$

$$\frac{\partial H_1}{\partial v_1} = \hat{v}_1 - u_1 + \frac{q_1^1}{2} - \delta_4^1 + \delta_6^1 = 0$$

$$\frac{\partial H_2}{\partial v_2} = 2(\hat{v}_2 - u_2) + q_2^2 - \delta_5^2 + \delta_7^2 = 0$$

$$\delta_i^j h_i(t, x^*, u_o, u, \hat{v})\ j = 1,2., \quad i = 0,1,\cdots,7.$$

$$\delta_i^j \geq 0\ j = 1,2., \quad i = 0,1,\cdots,7. \qquad (70)$$

solving the system of equations(70), it follows that $q_1^1(t) = 1$, $q_2^1(t) = 0$, $q_1^2(t) = 0$, $q_2^2(t) = \frac{2}{3}$ and the nash response set D is the set of all solutions $(u_0, u, \hat{v}) \in \Omega$ such that $\hat{v} = (\hat{v}_1, \hat{v}_2)$ where

$$\hat{v}_1 = \begin{cases} 0 & , \quad 0 \leq u_1 \leq \frac{1}{2} \\ u_1 - \frac{1}{2} & , \quad \frac{1}{2} < u_1 < \frac{3}{2} \\ 1 & , \quad u_1 \geq \frac{3}{2} \end{cases} \qquad (71)$$

and

$$\hat{v}_2 = \begin{cases} 0 & , \quad 0 \leq u_2 \leq \frac{1}{3} \\ u_2 - \frac{1}{3} & , \quad \frac{1}{3} < u_2 < 1 \\ \frac{2}{3} & , \quad u_2 \geq 1 \end{cases} \qquad (72)$$

now for the min-max solutions for the cooperative players against each player outside the coalition can be obtained by applying the necessary conditions (9)-(18)

$$\frac{\partial \bar{H}}{\partial u_1} = 2u_1^* + 2\bar{w}u_0 - 4\bar{w} - 2 - \bar{\varsigma}_1 + \bar{\varsigma}_3 = 0$$
$$\frac{\partial \bar{H}}{\partial u_2} = (1+\bar{w})u_2^* - (1-\bar{w})u_0 - (1+\bar{w}) - \bar{\varsigma}_2 + \bar{\varsigma}_3 = 0$$
$$\frac{\partial \bar{H}}{\partial v_1} = 2(1-\bar{w})(\bar{v}_1 - 2) + \frac{p_1}{2} - \bar{\eta}_4^1 + \bar{\eta}_6^1 = 0$$
$$\frac{\partial \bar{H}}{\partial v_2} = 2\bar{w}(\bar{v}_2 - 3) + p_2 - \bar{\eta}_5^2 + \bar{\eta}_7^2 = 0$$
$$\bar{\varsigma}_i h_i(t, x^*, u_o, u^*, \bar{v}) = 0, \qquad i = 0,1,\cdots,7.$$
$$\bar{\eta}_i^j h_i(t, x^*, u_o, u^*, \bar{v}) = 0, \qquad j = 1,2., \quad i = 0,1,\cdots,7.$$
$$\bar{\varsigma}_i \geq 0, \qquad i = 0,1,\cdots,7.$$
$$\bar{\eta}_i^j \leq 0, \qquad j = 1,2., \quad i = 0,1,\cdots,7. \qquad (73)$$

solving the system of equations (73), it yields that the pareto min-max solutions are $(u_0, u^*, \bar{v}) \in \Omega$ where $u^* \in P$ given by (68) and $\bar{v} = (0,0)$ for $\bar{w} \in (0,1)$. Furthermore, the hamiltonian function (38), and the necessary conditions equations(28)-(37) of the min-max solutions for the the first follower player outside the coalition with control v_1 yields the following:

$$\frac{\partial H_1}{\partial v_1} = v_1^* - \bar{u}_1^1 + \frac{q_1^1}{2} - \varsigma_4^1 + \varsigma_6^1 = 0$$
$$\frac{\partial H_1}{\partial u_1} = -v_1^* + 1 - \eta_1^1 + \eta_3^1 = 0$$
$$\frac{\partial H_1}{\partial u_2} = -\eta_2^1 + \eta_3^1 = 0$$
$$\frac{\partial H_2}{\partial v_2} = 2(v_2^* - \bar{u}_2^1) + q_2^2 - \varsigma_5^2 + \varsigma_7^2 = 0$$
$$\varsigma_i^j h_i(t, x^*, u_o, \bar{u}^1, v^*) = 0, \qquad j = 1,2., \quad i = 0,1,\cdots,7.$$
$$\varsigma_i^j \geq 0, \qquad j = 1,2., \quad i = 0,1,\cdots,7.$$
$$\eta_i^1 h_i(t, x^*, u_o, \bar{u}^1, v^*) = 0, \qquad i = 0,1,\cdots,7.$$
$$\eta_i^1 \leq 0, \qquad i = 0,1,\cdots,7. \qquad (75)$$

solving the equations(75); it follows that the min-max solution for the follower with control v_1 is given by $(u_0, \bar{u}^1, v^*) \in \Omega$ where $v_1^* = 1, \bar{u}_1^1 > \frac{3}{2}$,

$\overline{u}_2^1 < 4-\overline{u}_1^1$ and $v_2^* = v_2^*(u_2^1)$ given by (72). Similarly for player with controls v_2, equation(38), and the necessary conditions equations(28)-(37) gives

$$\frac{\partial H_2}{\partial v_2} = 2(v_2^* - \overline{u}_2^2) + q_2^2 - \varsigma_5^2 + \varsigma_7^2 = 0$$
$$\frac{\partial H_2}{\partial u_1} = -\eta_1^2 + \eta_3^2 = 0$$
$$\frac{\partial H_2}{\partial u_2} = -2(v_2^* - \overline{u}_2^2) + 1 - \eta_2^2 + \eta_3^2 = 0$$
$$\frac{\partial H_1}{\partial v_1} = v_1^* - \overline{u}_1^2 + \frac{q_1^1}{2} - \varsigma_4^1 + \varsigma_6^1 = 0$$
$$\varsigma_i^j h_i(t, x^*, u_o, \overline{u}^2, v^*) = 0, \qquad j = 1,2., \quad i = 0,1,\cdots,7.$$
$$\varsigma_i^j \geq 0, \qquad j = 1,2., \quad i = 0,1,\cdots,7.$$
$$\eta_i^2 h_i(t, x^*, u_o, \overline{u}^2, v^*) = 0, \qquad i = 0,1,\cdots,7.$$
$$\eta_i^2 \leq 0, \qquad i = 0,1,\cdots,7. \qquad (76)$$

where $i = 0,2,\cdots,7.$, solving the equations (76); it follows that the min-max solution for that player is given by $(u_0, \overline{u}^2, v^*) \in \Omega$ where $v_1^* = 0, v_2^* = \frac{2}{3}$ and $\overline{u}_1^2 = 0, \overline{u}_2^2 = 4$.

Finally the leader based on the optimality concept of the followers announce his control u_0 that minimize his cost, the corresponding hamiltonian function for the leader is given by (64)

$$\begin{aligned}
H_0(t,x,u_0,u,v,p_0) = {} & [(u_0-1)^2 + u_1 - u_2 - v_1 + v_2] + p_0^1(\frac{1}{2}v_1) + p_0^2(u_0 + v_2) \\
& + \overline{\lambda}_1[2u_1 + 2\overline{w}u_0 - 4\overline{w} - 2 - \hat{\delta}_1 + \hat{\delta}_3] \\
& + \overline{\lambda}_2[(1+\overline{w})u_2 - (1-\overline{w})u_0 - (1+\overline{w}) - \hat{\delta}_2 + \hat{\delta}_3] \\
& + \lambda_1[v_1 - u_1 + q_1^1/2 - \hat{\delta}_4^1 + \hat{\delta}_6^1] + \lambda_2[2(v_2 - u_2) + q_2^2 - \hat{\delta}_5^2 + \hat{\delta}_7^2] \\
& - \mu_0 u_0 - \mu_1 u_1 - \mu_2 u_2 - \mu_3(4 - u_1 - u_2) - \mu_4 v_1 - \mu_5 v_2 \\
& - \mu_6(1 - v_1) - \mu_7(\frac{2}{3} - v_2)
\end{aligned}$$

The necessary conditions for stakelberg solution with pareto, nash min-max followers (39)-(63), gives

$$\dot{p}_0^1(t) = -\frac{\partial H_0}{\partial x_1(t)} = 0$$
$$p_0^1(t_f) = \frac{\partial \phi_0(x(t_f))}{\partial x_1(t_f)} = 0$$

$$\dot{p}_0^2(t) = -\frac{\partial H_0}{\partial x_2(t)} = 0$$

$$p_0^2(t_f) = \frac{\partial \phi_0(x(t_f))}{\partial x_2(t_f)} = 0$$

$$\frac{\partial \overline{H}}{\partial u_1} = 2u_1^* + 2\overline{w}u_0^* - 4\overline{w} - 2 - \hat{\delta}_1 + \hat{\delta}_3 = 0$$

$$\frac{\partial \overline{H}}{\partial u_2} = (1+\overline{w})u_2^* - (1-\overline{w})u_0 - (1+\overline{w}) - \hat{\delta}_2 + \hat{\delta}_3 = 0$$

$$\frac{\partial H_1}{\partial v_1} = v_1^* - u_1^* + \frac{q_1^1}{2} - \hat{\delta}_4^1 + \hat{\delta}_6^1 = 0$$

$$\frac{\partial H_2}{\partial v_2} = 2(v_2^* - u_2^*) + q_2^2 - \hat{\delta}_5^2 + \hat{\delta}_7^2 = 0$$

$$\frac{\partial H_0}{\partial u_0} = 2(u_0^* - 1) + p_0^2 + 2\overline{w}\overline{\lambda}_1 - (1-\overline{w})\overline{\lambda}_2 - \mu_0 = 0$$

$$\frac{\partial H_0}{\partial u_1} = 1 + 2\overline{\lambda}_1 - \lambda_1 - \mu_1 + \mu_3 = 0$$

$$\frac{\partial H_0}{\partial u_2} = -1 + (1+\overline{w})\overline{\lambda}_2 - 2\lambda_2 - \mu_2 + \mu_3 = 0$$

$$\frac{\partial H_0}{\partial v_1} = -1 + \lambda_1 - \mu_4 + \mu_6 = 0$$

$$\frac{\partial H_0}{\partial v_2} = 1 + 2\lambda_2 - \mu_5 + \mu_7 = 0$$

$$\hat{\delta}_i h_i = 0 i = 0,1,\cdots,7.$$

$$\hat{\delta}_i^j h_i = 0 \, j = 1,2 \quad i = 0,1,\cdots,7.$$

$$\mu_i h_i = 0 i = 0,1,\cdots,7.$$

$$\hat{\delta}_i \geq 0 i = 0,1,\cdots,7.$$

$$\hat{\delta}_i^j \geq 0 \, j = 1,2 \quad i = 0,1,\cdots,7.$$

$$\mu_i \geq 0 i = 0,1,\cdots,7. \qquad (78)$$

Solving these system of equations (78) it yields that the stakelberg solution with pareto, nash min-max followers game is

$$(u_0^*, u_1^*, u_2^*, v_1^*, v_2^*) = (1, 1+\overline{w}, \frac{2}{1+\overline{w}}, 1, \frac{2}{3}) \text{ where } \overline{w} \in (\frac{1}{2}, 1)$$

Conclusion

This study tends to introduce a new approach to stackelberg dynamic game with various optimality response between the followers.

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