

# ANOTHER WEIGHTED WEIBULL DISTRIBUTION FROM AZZALINI'S FAMILY

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## Abstract

A new weighted Weibull distribution has been defined and studied. Some mathematical properties of the distribution have been studied and the method of maximum likelihood was proposed for estimating the parameters of the distribution. The usefulness of the new distribution was demonstrated by applying it to a real lifetime dataset.

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**Keywords:** Weighted Weibull, Order statistic, Maximum likelihood estimation, Azzalini

## Introduction

The Weibull distribution has received much attention in literature because of the advantage it has over other distributions in modeling lifetime data. However, researchers continue to develop different generalizations of the Weibull distribution to increase its flexibility in modeling lifetime data. Merovci and Elbatal (2015) developed the Weibull-Rayleigh distribution and demonstrated its application using lifetime data. Almalki and Yuan (2012) presented the new modified Weibull distribution by combining the Weibull and the modified Weibull distribution in a serial system. The hazard function of the newly proposed distribution is the sum of the Weibull hazard function and a modified Weibull hazard function.

Another generalization of the Weibull distribution is the exponentiated Weibull distribution of Mudholkar and Srivastava (1993). Mudholkar *et al.* (1995) and Mudholkar and Huston (1996) further studied the exponentiated Weibull distribution with some application to bus-motor failure data and flood. Pal *et al.* (2006) gave a re-introduction of the exponentiated Weibull distribution in more details. Al-Saleh and Agarwal (2006) proposed another extended version of the Weibull distribution. They demonstrated that the hazard function can exhibit unimodal and bathtub shapes. Xie and Lai (1996) developed the additive Weibull distribution with bathtub shaped hazard function obtained as the sum of two hazard functions

of the Weibull distribution. Zhang and Xie (2007) employed the Marshall and Olkin (1997) approach of adding new parameter to a distribution to propose the extended Weibull distribution. Lai *et al.* (2003) proposed a modification of the Weibull distribution by multiplying the Weibull cumulative hazard function by  $e^{\lambda x}$  and studied its properties.

In this article, a new generalization of the Weibull distribution based on a modified weighted version of Azzalini's (1985) approach has been proposed. If  $g_0(x)$  is a probability density function (pdf) and  $\bar{G}_0(x)$  is the corresponding survival function such that the cumulative distribution function (cdf),  $G_0(x)$ , exist; Then the new weighted distribution is defined as:

$$f(x; \alpha, \theta, \lambda) = K g_0(x) \bar{G}_0(\lambda x) \quad (1)$$

where  $K$  is a normalizing constant.

### Weighted Weibull Distribution

In this section, the density of the weighted Weibull distribution has been derived based on the definition given in equation (1). Consider a two parameter Weibull distribution with pdf given by:

$$g_0(x) = \alpha \theta x^{\theta-1} e^{(-\alpha x^\theta)}, \quad x > 0, \alpha > 0, \theta > 0 \quad (2)$$

The cdf is given by:

$$G_0(x) = 1 - e^{(-\alpha x^\theta)} \quad (3)$$

The survival function is given by:

$$\bar{G}_0(x) = e^{(-\alpha x^\theta)} \quad (4)$$

Using equations (1), (2) and (4) the pdf of the weighted weibull distribution is defined as:

$$f(x; \alpha, \theta, \lambda) = (1 + \lambda^\theta) \alpha \theta x^{\theta-1} e^{-(\alpha x^\theta + \alpha(\lambda x)^\theta)} \quad (5)$$

The corresponding cdf of the weighted Weibull distribution is given by:

$$F(x; \alpha, \theta, \lambda) = 1 - e^{-(\alpha x^\theta + \alpha(\lambda x)^\theta)} \quad (6)$$

where  $\alpha$  is a scale parameter and,  $\theta$  and  $\lambda$  are shape parameters. Figure 1 and 2 illustrates possible shapes of the pdf and the cdf of the weighted Weibull distribution for some selected values of the parameters  $\alpha$ ,  $\theta$  and  $\lambda$ .

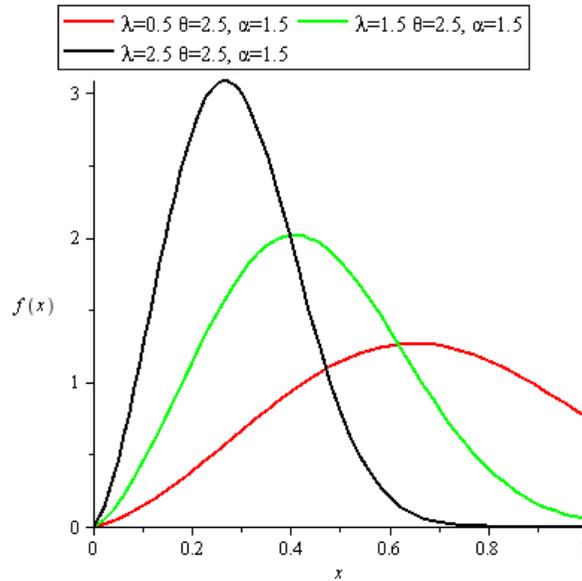


Figure 1: Probability density function of weighted Weibull distribution

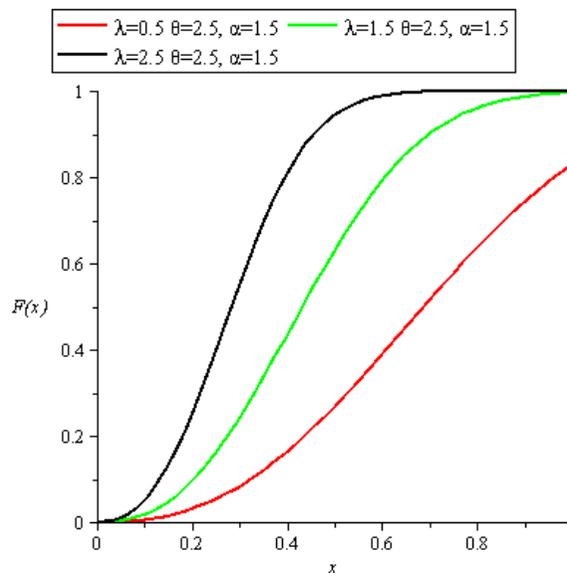


Figure 2: Cumulative distribution function of weighted Weibull distribution

The survival function is given by:

$$\bar{F}(x; \alpha, \theta, \lambda) = 1 - F(x; \alpha, \theta, \lambda) = e^{-(\alpha x^\theta + \alpha(\lambda x)^\theta)} \quad (7)$$

and the hazard function is:

$$h(x; \alpha, \theta, \lambda) = (1 + \lambda^\theta)\alpha\theta x^{\theta-1} \quad (8)$$

Figure 3 illustrates possible shapes of the hazard function of the weighted Weibull distribution for some selected values of the parameters  $\alpha$ ,  $\theta$  and  $\lambda$ .

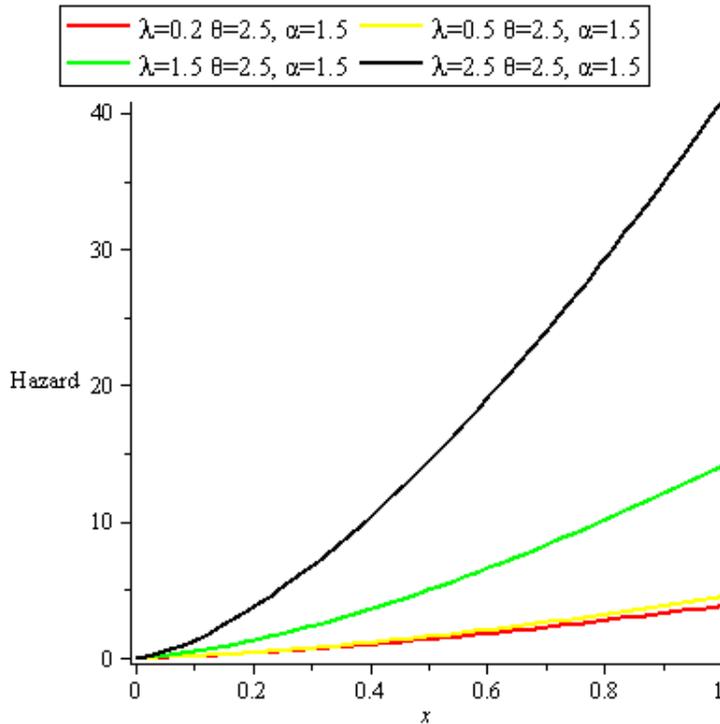


Figure 3: Hazard function of weighted Weibull distribution

**Statistical Properties**

In this section, the statistical properties of the weighted Weibull distribution is studied. The quantile function, skewness, kurtosis, mode, moment and moment generating function have been derived.

**Quantile function and Simulation**

Let  $Q(u), 0 < u < 1$  denote the quantile function for the weighted Weibull distribution. Then  $Q(u)$  is given by:

$$Q(u) = F^{-1}(u) = \left[ \frac{\ln\left(\frac{1}{1-u}\right)}{\alpha(1 + \lambda^\theta)} \right]^{\frac{1}{\theta}} \tag{9}$$

In particular, the distribution of the median is:

$$Q(0.5) = \left[ \frac{\ln 2}{\alpha(1 + \lambda^\theta)} \right]^{\frac{1}{\theta}}$$

To simulate from the weighted Weibull distribution is straight forward. Let  $u$  be a uniform variate on the unit interval  $(0,1)$ . Thus by means of the inverse transformation method, we consider the random variable  $X$  given by:

$$X = \left[ \frac{\ln \left( \frac{1}{1-u} \right)}{\alpha(1 + \lambda^\theta)} \right]^{\frac{1}{\theta}} \tag{10}$$

**Mode**

Consider the density of the weighted Weibull distribution given in (5). The mode is obtained by solving  $\frac{d \ln f(x)}{dx} = 0$  for  $x$ . Therefore the mode at  $x = x_0$  is given by:

$$x_0 = \left[ \frac{(\theta - 1)}{\alpha(1 + \lambda^\theta)} \right]^{\frac{1}{\theta}} \tag{11}$$

**Skewness and Kurtosis**

In this study, the quantile based measures of skewness and kurtosis was employed due to non-existence of the classical measures in some cases. The Bowley’s measure of skewness based on quartiles is given by:

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}$$

and the Moors’ kurtosis is on octiles and is given by:

$$M = \frac{Q(7/8) - Q(5/8) - Q(3/8) + Q(1/8)}{Q(6/8) - Q(2/8)}$$

where  $Q(.)$  represents the quantile function.

**Moment and Moment Generating Function**

In this section, the  $r^{th}$  non central moment and the moment generating function have been derived.

**Theorem 1.** If a random variable  $X$  has a weighted Weibull distribution, then the  $r^{th}$  non central moment is given by the following:

$$\mu'_r = \left[ \frac{1}{\alpha(1 + \lambda^\theta)} \right]^{\frac{r}{\theta}} \Gamma \left( 1 + \frac{r}{\theta} \right) \tag{12}$$

**Proof.**

$$\mu'_r = \int_0^\infty x^r f(x) dx$$

This implies

$$\mu'_r = \int_0^\infty x^r (1 + \lambda^\theta) \alpha \theta x^{\theta-1} e^{-(\alpha x^\theta + \alpha(\lambda x)^\theta)} dx \tag{13}$$

Let  $y = \alpha x^\theta + \alpha(\lambda x)^\theta, dy = \alpha \theta (1 + \lambda^\theta) x^{\theta-1} dx$  and  $x = \left[ \frac{y}{\alpha(1 + \lambda^\theta)} \right]^{\frac{1}{\theta}}$

$$\begin{aligned} \mu'_r &= \int_0^\infty \left[ \frac{y}{\alpha(1 + \lambda^\theta)} \right]^{\frac{r}{\theta}} e^{-y} dy \\ &= \left[ \frac{1}{\alpha(1 + \lambda^\theta)} \right]^{\frac{r}{\theta}} \int_0^\infty y^{(\frac{r}{\theta}+1)-1} e^{-y} dy \\ &= \left[ \frac{1}{\alpha(1 + \lambda^\theta)} \right]^{\frac{r}{\theta}} \Gamma\left(1 + \frac{r}{\theta}\right) \end{aligned} \tag{14}$$

This completes the proof.

If  $r = 1$ ,  $E(X) = \left[ \frac{1}{\alpha(1 + \lambda^\theta)} \right]^{\frac{1}{\theta}} \Gamma\left(1 + \frac{1}{\theta}\right)$

If  $r = 2$ ,  $E(X^2) = \left[ \frac{1}{\alpha(1 + \lambda^\theta)} \right]^{\frac{2}{\theta}} \Gamma\left(1 + \frac{2}{\theta}\right)$

Therefore the variance is given by

$$Var(X) = E(X^2) - \{E(X)\}^2$$

Theorem 2. Let  $X$  have a weighted Weibull distribution. The moment generating function of  $X$  denoted by  $M_X(t)$  is given by:

$$M_X(t) = \sum_{i=0}^\infty \frac{t^i}{i!} \left[ \frac{1}{\alpha(1 + \lambda^\theta)} \right]^{\frac{i}{\theta}} \Gamma\left(1 + \frac{i}{\theta}\right) \tag{15}$$

**Proof.**

By definition

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x) dx$$

Using Taylor series

$$\begin{aligned} M_X(t) &= \int_0^\infty \left( 1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \dots + \frac{t^nx^n}{n!} + \dots \right) f(x) dx \\ &= \sum_{i=0}^\infty \frac{t^i E(X^i)}{i!} \end{aligned}$$

$$= \sum_{i=0}^\infty \frac{t^i}{i!} \left[ \frac{1}{\alpha(1 + \lambda^\theta)} \right]^{\frac{i}{\theta}} \Gamma\left(1 + \frac{i}{\theta}\right)$$

This completes the proof.

**Renyi Entropy**

If  $X$  be a random variable having a weighted Weibull distribution. An important measure of the uncertainty of  $X$  is the Renyi entropy. The Renyi entropy is defined as:

$$I_R(\delta) = \frac{1}{1 - \delta} \log[I(\delta)]$$

where  $I(\delta) = \int_R [f(x)]^\delta dx$ ,  $\delta > 0$  and  $\delta \neq 1$ .

Theorem 3. For a random variable  $X$  having a weighted Weibull distribution, the Renyi entropy is given by:

$$I_R(\delta) = \frac{1}{1 - \delta} \log \left[ \theta(1 + \lambda^\theta)^\delta (\alpha\theta)^\delta \sum_{i=0}^{\infty} \frac{(-1)^i (\alpha\delta\lambda^\theta)^i (\alpha\delta)^{-(\delta\theta + \delta + i\theta + 1)}}{i!} \Gamma(\delta\theta - \delta + i\theta + 1) \right] \quad (16)$$

Where  $\Gamma(\cdot)$  is the gamma function.

### Reliability

Reliability of a component plays a significant role in Stress-Strength analysis of a model. If  $X$  is the strength and  $Y$  is the stress, the component fails when  $X \leq Y$ . Then the estimation of the reliability of the component  $R$  is  $\Pr(Y < X)$ .

$$R = \int_0^\infty f(x)F(x)dx = 1 - \int_0^\infty f(x)\bar{F}(x)dx$$

Theorem 4. If  $X$  is the strength and  $Y$  is the stress, then the reliability of the component  $R$  is given by:

$$R = 1 - \sum_{h=0}^{\infty} \frac{(-1)^h \alpha^h (1 + \lambda^\theta)^h [\alpha(1 + \lambda^\theta)]^{-h}}{h!} \Gamma(h + 1) \quad (17)$$

### Order Statistics

Let  $X_{(1)}$  denote the smallest of  $\{X_1, X_2, \dots, X_n\}$ ,  $X_{(2)}$  denote the second smallest of  $\{X_1, X_2, \dots, X_n\}$ , and similarly  $X_{(k)}$  denote the  $k^{th}$  smallest of  $\{X_1, X_2, \dots, X_n\}$ . Then the random variables  $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ , called the order statistics of the sample  $X_1, X_2, \dots, X_n$ , has probability density function of the  $k^{th}$  order statistic,  $X_{(k)}$ , as:

$$g_{k:n}(x) = \frac{n!}{(k - 1)! (n - k)!} f(x)[F(x)]^{k-1} [1 - F(x)]^{n-k}$$

for  $k = 1, 2, 3, \dots, n$ .

The pdf of the  $k^{th}$  order statistic is defined as:

$$g_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} (1 + \lambda^\theta) \alpha \theta x^{\theta-1} e^{-(\alpha x^\theta + \alpha(\lambda x)^\theta)} \times \left[1 - e^{-(\alpha x^\theta + \alpha(\lambda x)^\theta)}\right]^{k-1} \times \left[e^{-(\alpha x^\theta + \alpha(\lambda x)^\theta)}\right]^{n-k} \tag{18}$$

The pdf of the largest order statistic  $X_{(n)}$  is therefore:

$$g_{n:n}(x) = n(1 + \lambda^\theta) \alpha \theta x^{\theta-1} e^{-(\alpha x^\theta + \alpha(\lambda x)^\theta)} \times \left[1 - e^{-(\alpha x^\theta + \alpha(\lambda x)^\theta)}\right]^{n-1} \tag{19}$$

and the pdf of the smallest order statistic  $X_{(1)}$  is given by:

$$g_{1:n}(x) = n(1 + \lambda^\theta) \alpha \theta x^{\theta-1} e^{-(\alpha x^\theta + \alpha(\lambda x)^\theta)} \times \left[e^{-(\alpha x^\theta + \alpha(\lambda x)^\theta)}\right]^{n-1} \tag{20}$$

Theorem 5. The  $r^{th}$  non central moment of the  $k^{th}$  order statistics is given by:

$$\mu_r^{(k:n)} = \frac{n!}{(k-1)!(n-k)!} \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \binom{k-1}{j} \frac{(-1)^{j+l+m} (\alpha j + \alpha j \lambda^\theta)^l [\alpha(n-k) + \alpha \lambda^\theta(n-k)]^m}{l!m!} \times \left[\alpha(1 + \lambda^\theta)\right]^{-\left(\frac{r+l\theta+m\theta}{\theta}\right)} \Gamma\left(\frac{r+l\theta+m\theta}{\theta} + 1\right) \tag{21}$$

**Maximum Likelihood Estimation**

In this section, the method of maximum likelihood was considered for the estimation of the parameters of the weighted Weibull distribution. Consider a random sample of size  $n$ , consisting of value  $x_1, x_2, \dots, x_n$  from the weighted Weibull density

$$f(x; \alpha, \theta, \lambda) = (1 + \lambda^\theta) \alpha \theta x^{\theta-1} e^{-(\alpha x^\theta + \alpha(\lambda x)^\theta)}$$

The likelihood function of the above density is given by:

$$L(\mathbf{x}; \alpha, \lambda, \theta) = \prod_{i=1}^n \left[ (1 + \lambda^\theta) \alpha \theta x_i^{\theta-1} e^{-(\alpha x_i^\theta + \alpha(\lambda x_i)^\theta)} \right]$$

where  $\mathbf{x} = [x_1, x_2, \dots, x_n]'$ . The log-likelihood function is given by:

$$\ln L(\mathbf{x}; \alpha, \lambda, \theta) = n \ln(1 + \lambda^\theta) + n \ln \alpha + n \ln \theta + (\theta - 1) \sum_{i=1}^n \ln x_i - \alpha(1 + \lambda^\theta) \sum_{i=1}^n x_i^\theta \tag{22}$$

Taking the partial derivatives of the log-likelihood function in (22) with respect to the parameters  $\alpha, \lambda$  and  $\theta$  yields:

$$\frac{\partial \ln L(\mathbf{x}; \alpha, \lambda, \theta)}{\partial \alpha} = \frac{n}{\alpha} - (1 + \lambda^\theta) \sum_{i=1}^n x_i^\theta \tag{23}$$

$$\frac{\partial \ln L(\mathbf{x}; \alpha, \lambda, \theta)}{\partial \lambda} = \frac{n\theta\lambda^{\theta-1}}{(1 + \lambda^\theta)} - \alpha\theta\lambda^{\theta-1} \sum_{i=1}^n x_i^\theta \tag{24}$$

$$\begin{aligned} \frac{\partial \ln L(\mathbf{x}; \alpha, \lambda, \theta)}{\partial \theta} &= \frac{n\lambda^\theta \ln \lambda}{(1 + \lambda^\theta)} + \frac{n}{\theta} + \sum_{i=1}^n \ln x_i \\ &\quad - \alpha \sum_{i=1}^n [x_i^\theta \ln x_i + (\lambda x_i)^\theta \ln(\lambda x_i)] \end{aligned} \tag{25}$$

Setting equations (23), (24) and (25) to zero and solving them simultaneously yields the maximum likelihood estimates of the three parameters. By taking the second partial derivatives of (23), (24) and (25) the Fisher’s information matrix can be obtained by taking the negative expectations of the second partial derivatives. The inverse of the Fisher’s information matrix is the variance covariance matrix of the maximum likelihood estimators.

**Empirical Study**

In this section, an empirical study was carried out to investigate the effect of change in the values of the new parameter  $\lambda$  for  $\alpha = 1.5$  and  $\theta = 2.5$ . Table 1 provides the mean, variance, Bowley’s coefficient of Skewness and Moor’s coefficient of kurtosis. From Table 1, the mean and the variance of the weighted Weibull distribution decreases for an increase in the value of  $\lambda$ . However, the coefficient of skewness and kurtosis are not affected by increase in the value of  $\lambda$ .

Table 1: Mean, Variance, Skewness and Kurtosis of weighted Weibull distribution

$\lambda$	$\alpha = 1.5$		$\theta = 2.5$	
	Mean	Variance	Skewness	Kurtosis
0.2	0.7490938	0.1027476	0.03727047	0.1043721
0.3	0.7400435	0.1002798	0.03727047	0.1043721
0.4	0.7258902	0.09648081	0.03727047	0.1043721
0.5	0.7068685	0.09149059	0.03727047	0.1043721
1.0	0.5717475	0.059856	0.03727047	0.1043721
1.5	0.4443662	0.03615607	0.03727047	0.1043721
2.0	0.3534343	0.02287265	0.03727047	0.1043721
2.5	0.2903561	0.01543693	0.03727047	0.1043721
3.0	0.2452979	0.03727047	0.03727047	0.1043721

### Application

In this section, the application of the new weighted Weibull distribution is demonstrated using the lifetime data of 20 electronic components (see Murthy *et al.*, 2004, pp. 83,100). Teimouri and Gupta (2013) studied this data using a three-parameter Weibull distribution. In this study, the weighted Weibull distribution is fitted to this data and the results compared to that of Teimouri and Gupta (2013). The data is shown in Table 2. From Table 3, the Anderson-Darling (AD) statistics revealed that the weighted Weibull fits the data better than the three-parameter Weibull distribution.

Table 2: Lifetimes of 20 electronic components

0.03	0.22	0.73	1.25	1.52	1.8	2.38	2.87	3.14	4.72
0.12	0.35	0.79	1.41	1.79	1.94	2.4	2.99	3.17	5.09

Table 3: Estimated Parameters of dataset

Distribution	Estimated Parameters			AD Statistic
Weighted Weibull	$\alpha = 0.363$	$\theta = 1.196$	$\lambda = 0.233$	0.419
Three-parameter Weibull	$\alpha = 1.217$	$\beta = 2.072$	$\mu = -0.008$	0.432

### Conclusion

A new weighted Weibull distribution based on modified weighted version of Azzalini’s (1985) approach has been proposed. Some important and mathematical properties of the distribution have been derived. An empirical study was carried out to determine the effect of the new parameter on the mean, variance, skewness and kurtosis of the distribution. The application of the new distribution has been demonstrated using real life data. Future works include comparison of the new distribution with other modified weibull distributions, application of the distribution to censored dataset and comparison of different techniques for estimating the parameters of the distribution.

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