NEW NINTH– AND SEVENTH–ORDER METHODS FOR SOLVING NONLINEAR EQUATIONS

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Abstract

In this paper, we suggest and analyze some new higher-order iterative methods free from second derivative and used for solving of nonlinear equations. These methods based on a Halley iterative method and the weight combination of mid-point with Simpson quadrature formulas and using predictor–corrector technique. The convergence analysis of our methods is discussed. It is established that the new methods have convergence order nine and seven. Numerical tests show that the new methods are comparable with the well known existing methods and gives better results.

Keywords: Nonlinear equations, Convergence analysis, Higher order, Iterative methods, Halley iterative method

Introduction

Finding iterative methods for solving nonlinear equations is an important area of research in numerical analysis at it has interesting applications in several branches of pure and applied science can be studied in the general framework of the nonlinear equations $f(x)=0$. Due to their importance, several numerical methods have been suggested and analyzed under certain condition. These numerical methods have been constructed using different techniques such as Taylor series, homotopy perturbation method and its variant forms, quadrature formula, variational iteration method, and decomposition method. For more details, see [1-11]. In this paper, based on a Halley and the weight combination of mid-point with Simpson quadrature formulas and using predictor–corrector technique, we construct modification of Newton's method with higher-order convergence for solving nonlinear equations. The error equations are given theoretically to show that the proposed techniques have ninth - and seventh -order convergence. Commonly in the literature the efficiency of an iterative method is measured by the *efficiency index* defined as $I \approx p^{1/d}$ [12], where p is the order of convergence and d is the total number of functional evaluations per step. Therefore these methods have efficiency index $7^{1/6} \approx 1.383$ and $9^{1/7} \approx 1.368$ which are higher than $3^{1/4} \approx 1.3161$ of the DHM method [13]. Several examples are given to illustrate the efficiency and performance of these methods.

Iterative methods

Consider the nonlinear equation of the type

$$f(x) = 0 \tag{1}$$

For simplicity, we assume that *r* is a simple root of Eq. (1) and x_0 is an initial guess sufficiently close to *r*. Using the Taylor's series expansion of the function $f_k(x)$, we have

$$f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0) = 0 \tag{2}$$

First two terms of the equation (3) gives the first approximation, as

$$x = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{3}$$

This allows us to suggest the following one-step iterative method for solving the system of nonlinear equations (1).

Algorithm 2.1. For a given x_0, find the approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

which is the Newton method. It is well known that algorithm 2.1 has a quadratic convergence.

again from (2) we have

$$x = x_0 - \frac{f(x_0)}{f'(x_0) + \frac{1}{2}(x - x_0) f''(x_0)} \tag{4}$$

Substitution again of (4) into the right hand side of (3) gives the second approximation

$$x = x_0 - \frac{2f(x_0)f'(x_0)}{2[f'(x_0)]^2 - f(x_0)f''(x_0)}.$$

This formula allows us to suggest the following iterative methods for solving the nonlinear Eq. (1).

Algorithm 2.2. For a given x_0, compute approximates solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}.$$

This is known as Halley's method and has cubic convergence [6].

In the other hand we can write the differentiable function $f(x)$ as follows

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt. \tag{5}$$

If we approximate the integration in (1) with average of midpoint and Simpson quadrature formulas then we have

$$\int_{x_n}^{x} f'(t)dt = \left(\frac{x - x_n}{2}\right) f'\left(\frac{x + x_n}{2}\right) + \left(\frac{x - x_n}{12}\right)\left[f'(x) + 4f'\left(\frac{x + x_n}{2}\right) + f'(x_n)\right]. \tag{6}$$

From (5) and (6), we have

$$f(x) = f(x_n) + \left(\frac{x - x_n}{12}\right)\left[f'(x) + 10f'\left(\frac{x + x_n}{2}\right) + f'(x_n)\right]. \tag{7}$$

Since $f(x) = 0$ then

$$x = x_n - \frac{12f(x_n)}{f'(x) + 10f'\left(\frac{x + x_n}{2}\right) + f'(x_n)}. \tag{8}$$

With this fixed point formulation and with selecting *Predictor-Corrector of Newton* method we will have followed a two-step iterative method for solving the nonlinear equation (1) as follows

Algorithm 2.3. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{12f(x_n)}{[f'(x_n) + 10f'(w_n) + f'(y_n)]}, \qquad w_n = \frac{x_n + y_n}{2}$$

this Algorithm has cubic convergence [14].

Now using the technique of updating the solution, therefore, using Algorithm 2.3 as a predictor and Algorithm 2.2 as a corrector, we suggest and analyze a new three-step iterative methods for solving the nonlinear equation (1), which are the main motivation of this paper.

Algorithm 2.4. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = x_n - \frac{12f(x_n)}{\left[f'(x_n) + 10f'(w_n) + f'(y_n)\right]}, \qquad w_n = \frac{x_n + y_n}{2}$$

$$x_{n+1} = z_n - \frac{2f(z_n)f'(z_n)}{2f'^2(z_n) - f(z_n)f''(z_n)}.$$

Algorithm 2.4 is called the predictor-corrector Halley's method (PCH) and has ninth-order convergence. Per iteration of the iterative method 2.4 requires two evaluations of the function, four evaluations of first derivative, and one evaluations of second derivative. We take into account the definition of efficiency index [12], if we suppose that all the evaluations have the same cost as function one, we have that the efficiency index of the method 2.4 is $9^{1/7} \approx 1.368$ which is better $3^{1/4} \approx 1.316$ of the DHM method [13].

In order to implement Algorithm 2.4, one has to find the second derivative of this function, which may create some problems. To overcome this drawback, several authors have developed involving only the first derivative. This idea plays a significant part in developing some iterative methods free from second derivatives. The second derivative with respect to z, which may create some problems. To overcome this drawback, several authors have developed involving only the first derivatives. This idea plays a significant part in developing our new iterative methods free from second derivatives with respect to z. To be more precise, we now approximate $f''(z_n)$, to reduce the number of evaluations per iteration by a combination of already known data in the past steps. Toward this end, an estimation of the function $P_1(t)$ is taken into consideration as follows

$$P_1(t) = a + b(t - y_n) + c(t - y_n)^2 + d(t - y_n)^3$$

and also consider that this approximation polynomial satisfies the interpolation conditions $f(y_n) = P_1(y_n)$, $f(z_n) = P_1(z_n)$, $f'(y_n) = P_1'(y_n)$ and $f'(z_n) = P_1'(z_n)$. By substituting the known values in $P_1(t)$ we have a system of three linear equations with three unknowns. By solving this system and simplifying we have

$$f''(z_n) = \frac{2}{z_n - y_n}\left(2f'(z_n) + f'(y_n) - 3\frac{f(z_n) - f(y_n)}{z_n - y_n}\right) = P_1(z_n). \qquad (9)$$

then algorithm 2.4 can be written in the form of the following algorithm.

Algorithm 2.5. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = x_n - \frac{12f(x_n)}{[f'(x_n) + 10f'(\mathrm{w}_n) + f'(\mathrm{y}_n)]}, \qquad w_n = \frac{x_n + \mathrm{y}_n}{2}$$

$$x_{n+1} = z_n - \frac{2f(z_n)f'(z_n)}{2[f'(z_n)]^2 - f(z_n)P_1(z_n)}.$$

Algorithm 2.5 is called the predictor-corrector Modified Halley's method (PCMH1) and has ninth-order convergence. Per iteration of the iterative method 2.5 requires three evaluations of the function and four evaluations of first derivative. if we suppose that all the evaluations have the same cost as function one, we have that the efficiency index of the method 2.5 is $9^{1/7} \approx 1.368$ which is better $3^{1/4} \approx 1.316$ of the DHM method [13] and is same $9^{1/7} \approx 1.368$ of the method 2.4, but the main advantage of the method it's free from the second order derivative.

To be more precise, we now approximate $f'(z_n)$, to reduce the number of evaluations per iteration by a combination of already known data in the past steps. Toward this end, an estimation of the function $P_2(t)$ is taken into consideration as follows

$$P_2(t) = a + b(t - y_n) + c(t - y_n)^2$$
$$P_2'(t) = b + 2c(t - y_n)$$

By substituting in the known values

$$P_2(z_n) = f(z_n) = a + b(z_n - x_n) + c(z_n - y_n)^2$$
$$P_2'(z_n) = f'(z_n) = b + 2c(z_n - y_n)$$
$$P_2(y_n) = f(y_n) = a, \qquad P_2'(y_n) = f'(y_n) = b$$

we could easily obtain the unknown parameters. Thus we have

$$f'(z_n) = 2\left(\frac{f(z_n) - f(y_n)}{z_n - y_n}\right) - f'(y_n) = P_2(z_n) \qquad (10)$$

then algorithm 2.5 can be written in the form of the following algorithm.

Algorithm 2.6. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$z_n = x_n - \frac{12f(x_n)}{\left[f'(x_n)+10f'(\mathrm{w}_n)+f'(\mathrm{y}_n)\right]}, \qquad w_n = \frac{x_n+\mathrm{y}_n}{2}$$

$$x_{n+1} = z_n - \frac{2f(z_n)P_2(z_n)}{2[P_2(z_n)]^2 - f(z_n)P_1(z_n)}.$$

Algorithm 2.6 is called the predictor-corrector Modified Halley's method (PCMH2) and has seventh-order convergence. Per iteration of the iterative method 2.6 requires three evaluations of the function and three evaluations of first derivative. if we suppose that all the evaluations have the same cost as function one, we have that the efficiency index of the method 2.6 is $7^{1/6} \approx 1.383$ which is better than $9^{1/7} \approx 1.368$ of the method 2.4 and method 2.5.

Convergence analysis

Let us now discuss the convergence analysis of the above mentioned methods Algorithm 2. 3 and Algorithm 2. 4

Theorem 3.1 Let *r* be a sample zero of sufficient differentiable function $f :\subseteq R \to R$ for an open interval I . If x_0 is sufficiently close to r , then the two step method defined by our algorithm 2.3 has convergence is at least of order three.

Proof. Consider to

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{11}$$

$$x_{n+1} = x_n - \frac{12f(x_n)}{\left[f'(x_n)+10f'(\mathrm{w}_n)+f'(\mathrm{y}_n)\right]}, \qquad w_n = \frac{x_n+\mathrm{y}_n}{2} \tag{12}$$

Let r be a simple zero of f . Since f is sufficiently differentiable, by expanding $f(x_n)$ and $f'(x_n)$ about r , we get

$$f(x_n) = f(r) + (x_n - r)f'(r) + \frac{(x_n-r)^2}{2!}f^{(2)}(r) + \frac{(x_n-r)^3}{3!}f^{(3)}(r) + \frac{(x_n-r)^4}{4!}f^{(4)}(r) + \cdots,$$

then

$$f(x_n) = f'(r)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \cdots], \tag{13}$$

and

$$f'(x_n) = f'(r)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + \cdots], \tag{14}$$

where $c_k = \frac{1}{k!}\frac{f^{(k)}(r)}{f'(r)}, k = 1, 2, 3, \ldots$ and $e_n = x_n - r$.

Now from (13) and (14), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + \cdots, \tag{15}$$

From (11) and (15), we get

$$y_n = r + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + (-7c_2c_3 + 4c_2^3 + 3c_4)e_n^4 + \cdots, \tag{16}$$

From (16), we get,

$$f(y_n) = f'(r)[(y_n - r) + c_2(y_n - r)^2 + c_3(y_n - r)^3 + c_4(y_n - r)^4 + \cdots]$$

and

$$f'(y_n) = f'(r)[1 + 2c_2^2e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + (-11c_2^2c_3 + 8c_2^4 + 6c_2c_4)e_n^4 + \cdots].$$

Expanding $f'(w_n)$ about r, we get

$$f'(w_n) = f'(r)[1 + c_2e_n + (c_2^2 + \frac{3}{4}c_3)e_n^2 + (\frac{7}{2}c_2c_3 - 2c_2^3 + \frac{1}{2}c_4)e_n^3$$
$$+ (\frac{9}{2}c_2c_4 - \frac{29}{4}c_2^2c_3 + \frac{5}{16}c_5 + 4c_2^4 + 3c_3^2)e_n^4 + \cdots],$$

then

$$f'(x_n) + 10f'(w_n) + f'(y_n) = 12f'(r)[1 + c_2e_n + (\frac{7}{8}c_3 + c_2^2)e_n^2 + (\frac{13}{4}c_2c_3 + \frac{3}{4}c_4 - 2c_2^3)e_n^3$$
$$+ (\frac{5}{2}c_3^2 - \frac{53}{8}c_2^2c_3 + \frac{17}{4}c_2c_4 + \frac{65}{96}c_5 + 4c_2^4)e_n^4 + \cdots]$$

From (12), $e_{n+1} = x_{n+1} - r$ and $e_n = x_n - r$ then we will have

$$e_{n+1} = (c_2^2 - \frac{1}{8}c_3)c_3e_n^3 + O(e_n^4) \tag{17}$$

which shows that Algorithm 2.3 is at least a third order convergent method, the required result.

Theorem 3.2 Let r be a sample zero of sufficient differentiable function $f :\subseteq R \rightarrow R$ for an open interval I . If x_0 is sufficiently close to r , then the two step method defined by our algorithm 2.4 has convergence is at least of order nine.

Proof. Consider to

$$x_{n+1} = z_n - \frac{2f(z_n)f'(z_n)}{2[f'(z_n)]^2 - f(z_n)P_1(z_n)}. \tag{18}$$

Again by using Taylor's expansion we can get

$$f(z_n) = f'(r)[Z + c_2 Z^2 + c_3 Z^3 + \cdots], \tag{19}$$

$$f'(z_n) = f'(r)[1 + 2c_2 Z + 3c_3 Z^2 + 4c_4 Z^3 + \cdots], \tag{20}$$

and

$$f''(z_n) = f'(r)[2c_2 + 6c_3 Z + 12c_4 Z^2 + \cdots] \tag{21}$$

The equation (16) can be written as the form

$$x_{n+1} = z_n - f(z_n)\left(\frac{f'(z_n)}{2[f'(z_n)]^2 - f(z_n)f''(z_n)}\right) \tag{22}$$

$$2f'^2(z_n) - f(z_n)f''(z_n) = f'^2(r)[2 + 6c_2 Z + (6c_3 + 6c_2^2) Z^2 + (4c_4 + 16c_2c_3) Z^3 + \cdots] \tag{23}$$

Using (19), and (23) in (22), we have

$$x_{n+1} = r + (c_2^2 - c_3) Z^3 + (c_2c_3 - c_4) Z^4 + \cdots \tag{24}$$

from (17) $Z = y_n - r = (c_2^2 - \frac{1}{8}c_3)e_n^3 + O(e_n^4)$

$$x_{n+1} = r + \frac{1}{512}(8c_2^2 - c_3)^3(c_2^2 - c_3)e_n^9 + O(e_n^{10})$$

or

$$e_{n+1} = \frac{1}{512}(8c_2^2 - c_3)^3(c_2^2 - c_3)e_n^9 + O(e_n^{10})$$

which shows that Algorithm 2.3 has ninth- order of convergence.

In Similar way, we observe that the PCMH1 and PCMH2 have the error equations are given as follows respectively

$$e_{n+1} = \frac{1}{512}(8c_2^2 - c_3)^3(c_2^2 - c_3)e_n^9 + O(e_n^{10}),$$

$$e_{n+1} = \frac{1}{8}(c_3 - 8c_2^2)c_2^2c_3e_n^7 + O(e_n^8).$$

Numerical examples

For comparisons, we have used the fourth-order Jarratt method [15] (JM) and Ostrowski's method (OM) [12] defined respectively by

$$y_n = x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = x_n - \left(1 - \frac{3}{2}\frac{f'(y_n) - f'(x_n)}{3f'(y_n) - f'(x_n)}\right)\frac{f(x_n)}{f'(x_n)}$$

and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = y_n - \frac{f(x_n)}{f(x_n) - 2f(y_n)}\frac{f(y_n)}{f'(x_n)}.$$

We consider here some numerical examples to demonstrate the performance of the new modified iterative methods, namely (PCH), (PCMH1) and (PCMH2). We compare the classical Newton's method (NM), Jarratt method (JM), the Ostrowski's method (OM) (PCH), (PCMH1) and (PCMH2), in this paper. In the Tables 1, 2 the number of iteration is $n = 3$ for all our examples. But in Table 3 our examples are tested with precision $\varepsilon = 10^{-200}$. The following stopping criteria is used for computer programs: $|x_{n+1} - x_n| + |f(x_{n+1})| < \varepsilon$. And the computational order of convergence (*COC*) can be approximated using the formula,

$$COC \approx \frac{\ln|(x_{n+1} - x_n)/(x_n - x_{n-1})|}{\ln|(x_n - x_{n-1})/(x_{n-1} - x_{n-2})|}$$

Table 1, we listed the number of iterations for various methods. Tables 1, 2, shows the difference of the root r and the approximation x_n to r, where r is the exact root computed with 2000 significant digits, but only 25 digits are displayed for x_n. The absolute values of the function $f(x_n)$ and the computational order of convergence (*COC*) are also shown in Tables 1, 2. All the computations are performed using Maple 15. The following examples are used for numerical testing:

$f_1(x) = x^3 + 4x^2 - 10, \quad x_0 = 1.$ $\qquad f_2(x) = \sin^2 x - x^2 + 1, \quad x_0 = 1.3.$

$f_3(x) = x^2 - e^x - 3x + 2, \quad x_0 = 2.$ $\qquad f_4(x) = \cos x - x, \quad x_0 = 1.7.$

$f_5(x) = (x-1)^3 - 1, \quad x_0 = 2.5.$ $\qquad f_6(x) = x^3 - 10, \quad x_0 = 2.$

$f_7(x) = e^{x^2 + 7x - 30} - 1, \quad x_0 = 3.1.$

Table 1. Comparison of different methods

Method	x_0	x_3	*COC*	$\lvert x_3 - x_2 \rvert$	$\lvert f(x_3) \rvert$
f_1	1				
NM		1.3652366002021159462369662	1.88	3.66E-03	1.09E-04
JM		1.3652300134140968457610286	4.10	4.50E-12	5.95E-46
OM		1.3652300134140968457610286	4.10	4.50E-12	5.95E-46
PCH		1.3652300134140968457608068	9.09	0.10E-56	0.43E-514
PCNH1		1.3652300134140968457608068	9.10	0.10E-56	0.43E-514
PCNH2		1.365230013414096847608068	7.08	0.27E-38	0.63E-271
f_2	1.3				
NM		1.4044916527111965739297374	1.98	7.57E-05	1.12E-08
JM		1.4044916482153412260350868	4.03	5.09E-18	6.61E-70
OM		1.4044916482153412260350868	4.03	5.96E-18	1.29E-69
PCH		1.4044916482153412260350868	9.03	0.36E-86	0.31E-778
PCNH1		1.4044916482153412260350868	9.03	0.49E-86	0.43E-777
PCNH2		1.4044916482153412260350868	7.02	0.19E-58	0.76E-412
f_3	2				
NM		0.2575292578013089584442857	7.68	3.31E-03	3.88E-06
JM		0.2575302854398607604553673	4.35	6.21E-06	3.44E-23
OM		0.2575302854398607604553673	4.55	8.79E-06	1.02E-22
PCH		0.2575302854398607604553673	9.57	0.59E-52	0.64E-479
PCNH1		0.2575302854398607604553673	9.43	0.42E-56	0.31E-516
PCNH2		0.2575302854398607604553673	7.57	0.93E-25	0.18E-180
f_4	1.7				
NM		0.7390851658032147634513238	1.53	3.84E-04	5.45E-08
JM		0.7390851332151606416553121	3.66	1.47E-12	1.85E-49
OM		0.7390851332151606416553121	3.67	3.34E-12	5.32E-48
PCH		0.7390851332151606416553121	8.73	0.14E-63	0.59E-579
PCNH1		0.7390851332151606416553121	8.73	0.12E-63	0.26E-579
PCNH2		0.7390851332151606416553121	6.67	0.97E-44	0.26E-311

Table 2. Comparison of different methods

Method	x_0	x_3	*COC*	$\lvert x_3 - x_2 \rvert$	$\lvert f(x_3) \rvert$
f_5	2.5				
NM		2.0003266792741527249601052	1.98	1.80E-02	9.80E-04
JM		2	3.73	2.55E-08	8.43E-31
OM		2	3.73	2.55E-08	8.43E-31
PCH		2	8.69	0.51E-37	0.41E-335
PCNH1		2	8.69	0.51E-37	0.41E-335
PCNH2		2	6.69	0.12E-25	0.28E-181
f_6	2				
NM		2.1544346922369133091005011	1.97	6.89E-05	3.07E-08
JM		2.1544346900318837217592936	4.02	2.71E-19	4.98E-75
OM		2.1544346900318837217592936	4.02	2.71E-19	4.98E-75
PCH		2.1544346900318837217592936	9.02	0.92E-93	0.80E-839
PCNH1		2.1544346900318837217592936	9.02	0.92E-93	0.80E-839
PCNH2		2.1544346900318837217592936	7.02	0.19E-58	0.35E-412
f_7	3.1				
NM		3.0007511637578020952127918	2.24	1.02E-02	9.81E-03
JM		3	3.91	1.46E-07	6.17E-25
OM		3	3.92	9.81E-08	1.12E-25
PCH		3	8.71	0.87E-29	0.33E-254
PCNH1		3	8.74	0.50E-28	0.21E-247
PCNH2		3	6.65	0.14E-15	0.82E-105

Table 3. Comparison of Number of iterations for various methods.

Method	f_1	f_2	f_3	f_4	f_5	f_6	f_7
Guess	1	1.3	2	1.7	2.5	2	3.1
NM	12	11	12	11	13	11	13
JM	7	6	7	7	7	6	7
OM	7	6	7	6	7	6	7
PCH	5	5	5	5	5	5	5
PCMH1	4	4	4	4	4	4	4
PCMH2	4	4	4	4	4	4	5

Results are summarized in Table 1, 2 and Table 3 as it shows, new algorithms are comparable with all of the methods and in most cases gives better or equal results.

Conclusions

In this paper, we have suggested and analyzed some new higher-order iterative methods and used for solving of nonlinear equations. These methods based on a Halley iterative method and the weight combination of mid-point with Simpson quadrature formulas and using predictor–corrector technique. The error equations are given theoretically to show that the proposed techniques have ninth- and seventh-order convergence. The new methods attain efficiency indices of 1.383 and 1.368, which makes them competitive. In addition, the proposed methods have been tested on a series of examples published in the literature and show good results when compared it with the previous literature.

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