NUMERICAL SOLUTION OF THIN FILM EQUATION IN A CLASS OF DISCONTINUOUS FUNCTIONS

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Abstract

In this paper, an original method has been suggested to find a numerical solution of initial value problem for a fourth order degenerate diffusion equation which models the thin film flow. For this, an auxiliary problem established in a special way and having some advantages over the main problem has been introduced. Advantages of the auxiliary problem allow us to apply one of the well-known methods in literature, and thus the numerical solution of the main problem can be calculated by using the obtained solution.

Keywords: Thin film equation, Weak solution, Numerical solution in a class of discontinuous functions

Introduction

Let R^2 be an Euclidean space of points (x,t) and let $G \subset R^2$ be a rectangular region as $G = I \times [0,T)$, where $I = [-a,a]$ and a, T are given constants.

In G, we consider the fourth-order double degenerate nonlinear thin film equation as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u \frac{\partial^3 u}{\partial x^3} \right)$$

$$(1)$$

with the following initial

$$u(x,0) = u_0(x) \qquad\qquad (2)$$

and boundary

$$\frac{\partial^k u(\pm a,t)}{\partial x^k} = 0, \ (k = 0,1) \tag{3}$$

conditions. Here, the function $u_0(x)$ describes the finite mass, therefore, $u_0(x) \geq 0$ and the boundary conditions show that the fluid is permitted to drain over the edges $x = \pm a$.

An analysis of the solution obtained in [5], shows that $u(x,t)$, $\frac{\partial u(x,t)}{\partial x}$, $u(x,t)\frac{\partial u(x,t)}{\partial x}$, $u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}$ and $u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3} \to 0$, but $\frac{\partial^2 u(x,t)}{\partial x^2}$, $\frac{\partial^3 u(x,t)}{\partial x^3}$ for $x \to 1^-$. On the basis of these estimates we can say that problem (1)-(3) does not have a classical solution.

Since equation (1) is degenerated at $u(x,t) = 0$, following [1] we consider the approximating equation as

$$\frac{\partial u^\varepsilon}{\partial t} = -\frac{\partial}{\partial x}\left((u^\varepsilon + \varepsilon)\frac{\partial^3 u^\varepsilon}{\partial x^3}\right) \quad \text{in } Q_T, \tag{4}$$

where ε is a positive parameter. In this situation, it is also necessary to approximate $u_0(x)$ in $H^1(\Omega)$– norm by $C^{4+\alpha}$ functions $u_0^\varepsilon(x)$ satisfying the conditions (3), and replacing (2).

$$u(x,0) = u_0^\varepsilon(x). \tag{5}$$

Under the assumption that $u^\varepsilon(x,t)$ is a solution of problem (4), (5) in Q_σ, for some $0 < \sigma < T$ we derive estimates to be used later. Putting instead of u the u^ε we begin with

$$\int_I\left[\left(\frac{\partial u(x,t+\tau)}{\partial x}\right)^2 - \left(\frac{\partial u(x,t)}{\partial x}\right)^2\right]dx =$$

$$-\int_I\left[\frac{\partial^2 u(x,t+\tau)}{\partial x^2} + \frac{\partial^2 u(x,t)}{\partial x^2}\right]\left[\frac{\partial u(x,t+\tau)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right]dx,$$

since $\frac{\partial u(x,t)}{\partial x} = 0$ on the boundary. Dividing by τ and letting $\tau \to 0$ we get for any $0 < t_1 < t_2 < \sigma$

$$-\int_{t_1}^{t_2}\int_I \frac{\partial u(x,t)}{\partial t}\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{2}\int_I\left(\frac{\partial u(x,t)}{\partial x}\right)^2 dx\,|_{t_1}^{t_2}\ .$$

Multiplying (4) by $\frac{\partial^2 u^\varepsilon}{\partial x^2}$ and integrating over $Q_T (0 < T < \sigma)$ and using the last identity, we obtain

$$\frac{1}{2}\int_I\left(\frac{\partial u^\varepsilon(x,T)}{\partial x}\right)^2 dx \le \frac{1}{2}\int_I\left(\frac{\partial u^\varepsilon(x,0)}{\partial x}\right)^2 dx.$$

Now, integrating (4) over Ω_T we also obtain

$$\int_I\left(\frac{\partial u^\varepsilon(x,T)}{\partial x}\right)^2 dx \le \int_I\left(\frac{\partial u^\varepsilon(x,0)}{\partial x}\right)^2 dx. \qquad (6)$$

Integrating (4) over Ω_T we obtain

$$\int_I u^\varepsilon(x,T)dx \le \int_I u^\varepsilon(x,0)dx. \qquad (7)$$

Auxiliary Problem

Integrating equation (1) with respect to x from $-a$ until to x and using condition (3), we obtain

$$\frac{\partial}{\partial t}\int_{-a}^x u(\xi,t)d\xi = u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3} - u(-a,t)\frac{\partial^3 u(-a,t)}{\partial x^3}. \qquad (8)$$

Taking into consideration the equality

$$u(x,t)\frac{\partial^3 u(x,t)}{\partial x^3} = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}\right) - \frac{1}{2}\left[\frac{\partial}{\partial x}\left(\frac{\partial u(x,t)}{\partial x}\right)^2\right]$$

and once more integrating equation (8) with respect to x from $-a$ until to x, and compensating relation of integration to zero, we get

$$\frac{\partial}{\partial t}\int_{-a}^x\int_{-a}^x u(\xi,t)d\xi d\xi = u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} - u(-a,t)\frac{\partial^2 u(-a,t)}{\partial x^2}$$

$$-\left[\frac{1}{2}\left(\frac{\partial u(x,t)}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial u(-a,t)}{\partial x}\right)^2\right]. \qquad (9)$$

In accordance with

$$u(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right) - \left(\frac{\partial u(x,t)}{\partial x}\right)^2,$$

from (9) it follows that

$$\frac{\partial}{\partial t}\int_{-a}^x\int_{-a}^x\int_{-a}^x u(\xi,t)d\xi d\xi d\xi = u(x,t)\frac{\partial u(x,t)}{\partial x} - u(-a,t)\frac{\partial u(-a,t)}{\partial x}$$

$$-\frac{3}{2}\int_{-a}^x\left(\frac{\partial u(\xi,t)}{\partial\xi}\right)^2 d\xi. \qquad (10)$$

Integrating again in last formula (10), we have

$$\frac{\partial}{\partial t}\int_{-a}^{x}\int_{-a}^{x}\int_{-a}^{x}\int_{-a}^{x}u(\xi,t)d\xi d\xi d\xi d\xi = \frac{1}{2}u^2(x,t)-\frac{1}{2}u^2(-a,t)$$

$$\frac{3}{2}\int_{-a}^{x}\int_{-a}^{x}\left(\frac{\partial u(\xi,t)}{\partial \xi}\right)^2 d\xi d\xi. \qquad (11)$$

Using the Cauchy formula, we get

$$\frac{1}{3!}\frac{\partial}{\partial t}\int_{-a}^{x}(x-\eta)^3 u(\eta,t)d\eta = \frac{1}{2}u^2(x,t)-\frac{3}{2}\int_{-a}^{x}(x-\eta)\left(\frac{\partial u(\eta,t)}{\partial \eta}\right)^2 d\eta.$$

(12)

It is clear that if the functions $u(x,t)$ and $\dfrac{\partial u(x,t)}{\partial x}$ are differentiable continuous then the equations (12) or (11) and (1) are equivalent. By differentiating four times the last equation with respect to x, we prove this claim.

Numerical Algorithm

To approximate of equation (12) by the finite difference formulas, at first we cover the domain G by the grid $\Omega_{h_x h_t} = \Omega_{h_x} \times \Omega_{h_t}$. Here

$$\Omega_{h_x} = \left\{x_i \big| x_i = a + ih_x, i = 0,1,...,n\right\} \text{ and } \Omega_{h_t} = \left\{t_k \big| t_k = kh_t, k = 0,1,2,...\right\}.$$

The number $h_x = \dfrac{a}{n}$, $h_t > 0$ and h_t will be obtained from the condition of stability of the difference scheme.

Now, we construct a sub grid in Ω_{h_x}. For any i, we cover the interval $[x_i, x_{i+1}]$ with

$$\Omega_{h_\xi} = \left\{\xi_v \big| \xi_v = a + vh_\xi, v = 0,1,...,np\right\}.$$

Here $p > 0$ is any constant. As it is seen, $\xi_{pi} = x_i$. Thus we get $\Omega_{h_\xi h_t} \subseteq \Omega_{h_x h_t}$. We now may present a finite difference scheme to equation (12). At first, using cubature formulas for example, the method of rectangles, the integrals

$$\int_{-a}^{x}(x-\xi)^3 U(\eta,t)d\eta \text{ and } \int_{-a}^{x}(x-\eta)\left(\frac{\partial u}{\partial \eta}\right)^2 d\eta \text{ are approximated as follows}$$

$$\int_{-a}^{x_i}(x_i-\xi)^3 U(\eta,t)d\eta \cong h_\xi \sum_{v=1}^{pi}(x_i-\tilde{\eta}_v)^3 U(\tilde{\eta}_v,t_k), \qquad (13)$$

$$\int_{-a}^{x_i}(x_i-\eta)\left(\frac{\partial u}{\partial \eta}\right)^2 d\eta \cong \frac{1}{h_\xi}\sum_{v=1}^{pi}(x_i-\tilde{\eta}_v)[U(\tilde{\eta}_v,t_k)-U(\tilde{\eta}_{v-1},t_k)]^2 \qquad (14)$$

where $\widetilde{\eta}_\nu = \dfrac{\xi_\nu + \xi_{\nu+1}}{2}$. Taking into consideration expressions (13) and (14), since $x_i - \widetilde{\eta}_{pi} \neq 0$ for any i, integro-differential equation (12) can be approximated by the finite difference as follows

$$U(\widetilde{\eta}_{pi}, t_{k+1}) = (x_i - \widetilde{\eta}_{pi})^{-3}\left\{ -\sum_{\nu=1}^{pi-1}(x_i - \widetilde{\eta}_\nu)^3 U(\widetilde{\eta}_\nu, t_{k+1}) + \sum_{\nu=1}^{pi}(x_i - \widetilde{\eta}_\nu)^3 U(\widetilde{\eta}_\nu, t_k) + \frac{3h_t}{h_\xi}U^2(x_i t_k) \right.$$

$$\left. -\frac{9h_t}{h_\xi^2}\sum_{\nu=1}^{pi}(x_i - \widetilde{\eta}_\nu)\left[U(\widetilde{\eta}_\nu, t_k) - U(\widetilde{\eta}_{\nu-1}, t_k)\right]^2 \right\} \tag{15}$$

$(i = 0,1,2,...)$, where U_i, U_{i-1} and $\hat{U}_i$ are approximate values of the function $u(x,t)$ at any point (x_i, t_k), (x_{i-1}, t_k) and $(x_i, t_k + \tau)$ of the grid $\Omega_{\tau,h}$, respectively.

The initial and boundary conditions are

$$U_{i0} = u_0(x_i),$$

$$\left.\frac{U_{i+1} - U_i}{h}\right|_{i=n-1} = 0.$$

Note: To realize of our algorithm (14), at first, $\hat{U}_0$ is found at point x_0 by using Euler's method, then the unknown values are found time level $t_k = (k+1)\tau$, $(k = 0,1,2,...)$ from algorithm (15). The coefficients in (15) and other initial functions are calculated in time level t_k.

Now we will investigate the consistence and convergence of difference scheme (14) to solution (12).

Let $\varepsilon_{i,k}$, $\eta_{i,k}$ are the errors of approximation by the cubature formula of the integrals involving equation (12) by finite difference formulas.

Otherwise, let $\delta_{i,k}$, $\omega_{i,k}$ are errors of approximation of $\dfrac{\partial w(x,t)}{\partial t}$ and $\dfrac{\partial u(x,t)}{\partial x}$ by finite difference formulas, respectively.

Here

$$w(x,t) = \int_{-a}^{x}(x-\eta)^3 u(\eta, t)d\eta$$

and

$$\varepsilon_{i,k} = w(x,t) - h\sum_{j=1}^{m\cdot i}(x_i - \eta_i)^3 U_j,$$

$$\eta_{i,k} = \int_{-a}^{x}(x-\eta)\left(\frac{\partial u(\eta,t)}{\partial \eta}\right)^{2}d\eta - h\sum_{j=1}^{i}(x_{i}-\eta_{i})\left[\frac{U_{j+1}-U_{j}}{h}+\omega_{j,k}\right],$$

$$\delta_{i,k} = \frac{\partial w(x,t)}{\partial t} - \frac{\hat{W}_{i}-W_{i}}{\tau},$$

$$\omega_{i,k} = \frac{\partial u(x,t)}{\partial x} - \frac{U_{i+1}-U_{i}}{h}.$$

Due to the fact that functions $u(x,t)$ and $\dfrac{\partial u(x,t)}{\partial x}$ are continuous, $\omega_{i,k} \to 0$, i.e.

$$\omega_{i,k} = \frac{\partial u(x_{i},t_{k})}{\partial x} - \frac{U_{i+1}-U_{i}}{h} = \frac{\partial u(x_{i},t_{k})}{\partial x} - \frac{\partial u(x_{i}^{*},t_{k})}{\partial x} = \pi\left(\frac{\partial u(x,t)}{\partial x}\right) \to 0,$$

(16)

$x_{i}^{*} \in (x_{i},x_{i+h})$, where $\pi(f)$ is modulus continuity of any function $f(x)$ on any interval $[-a,a]$, that is,

$$\pi(f) = \sup_{|t-x|<h}|f(t)-f(x)|.$$

Let $\varphi(x,t) = (x-\eta)^{3}u(x,t)$. It is easy to see that the functions $\varphi(x,t)$ and $\varphi'(x,t)$ are continuous. Then, for $\varepsilon_{i,k}$ we have

$$\varepsilon_{i,k} = ah\varphi'(z,t)$$

(17)

for some $z \in [-a,a]$. Let

$$\psi(x,t) = (x-\eta)\left(\frac{\partial u(x,t)}{\partial x}\right)^{2}.$$

Due to the fact that the function $\psi(x,t)$ is continuous for $\eta_{i,k}$, we get

$$\eta_{i,k} = \int_{-a}^{x}\psi(\eta,t)d\eta - h\sum_{j=1}^{i}(x_{i}-\eta_{i})\frac{U_{j+1}-U_{j}}{h}$$

$$-h\sum_{j=1}^{i}(x_{i}-\eta_{i})\omega_{j,k} = ah\psi'(z,t) - h\sum_{j=1}^{i}(x_{i}-\eta_{i})\omega_{j,k} \qquad (18)$$

$$\delta_{i,k} = \frac{1}{3!}\frac{\partial w(x_{i},t_{k})}{\partial t} - \frac{1}{3!}\frac{\hat{W}_{i}-W_{i}}{\tau}$$

$$= \frac{1}{2}U^{2}(x_{i},t_{k}) - \frac{3}{2}\int_{-a}^{x_{i}}\psi(\eta,t)d\eta - \left[\frac{1}{2}U_{i}^{2} - \frac{3}{2}h\sum_{j=1}^{i}\psi_{j}\right]$$

$$= \frac{1}{2}\left(U^2(x_i,t_k) - U_j^2\right) - \left[\frac{3}{2}\int_{-a}^{x_i}\psi(\eta,t)d\eta - \frac{3}{2}h\sum_{j=1}^{i}\psi_j\right] = \max_{(x_i,t_k)} u.\pi(u) - \frac{3}{2}\eta_{i,k}.$$

(19)

As it is seen from (16)-(19) it follows that difference scheme (14) is consist to (12)

Numerical Experiments

In order to test the proposed method, we have used the data from paper [5]. The integral of u_0 is calculated as follows

$$u_0 = \begin{cases} (a - \Delta x - x)^3(a - \Delta x + x)^3, & |x| \le a - \Delta x, \\ \\ 0, & |x| > a - \Delta x, \end{cases}$$

(20)

$$\int_{-a}^{a} u_0(x)dx = \lim_{\Delta x \to 0}\left\{\int_{-a+\Delta x}^{0} u_0(x)dx + \int_{0}^{a-\Delta x} u_0(x)dx\right\} = \frac{12a^7}{7}.$$

Using algorithm (15), within the limit of initial condition (20) some computer experiments are carried out. As it is seen, obtained results approach sufficiently enough to exact solution, giving in paper [5]. Theoretical investigation of convergence and stability of finite difference scheme (15) will be a matter of next research.

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