COMPATIBLE TOTAL ORDERS ON $BR(G,\theta)$, WHERE G IS THE GROUP OF POLYNOMIALS WITH USUAL ADDITION

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Abstract:
In this paper we define a compatible total order on the set of all polynomials with real coefficients and finite degree. We use the order defined on the Bruck-Reilly extension $BR(G,\theta)$ for that purpose.

Keywords: Bruck-Reilly extension, Compatible total order, Polynomials

Introduction

We shall assume that the reader is familiar with the basic definitions and results concerning semigroups. In particular, we shall assume familiarity with the basic results on regular and inverse semigroups. These results and other undefined terminology can be found in [2].

Let G be a group and θ an endomorphism of G. Let $BR(G,\theta)$ be the Bruck-Reilly extension of G defined by θ. The elements of $BR(G,\theta)$ have the form $b^r g a^s$, with $g \in G$ and $r, s \in I\!N^0$. The multiplication is defined by
$b^r g a^s . b^u h a^v = b^{u \vee s - s + r} \theta^{u \vee s - s}(g) \theta^{s \vee u - u}(h) a^{s \vee u - u + v}$,
with $\theta(g) = agb,$ for all $g \in G$.

The relation $\leq_G$

Let $P(x) = \{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,\ a_n \neq 0\}$ be the set of all polynomials with real coefficients and finite degree. The set $P(x)$ with the usual addition is a group that we will call G. Define the function $\theta : P(x) \to \Re$, that transforms each $f \in P(x)$ in $\beta(f) = a_n$, with $a_n \neq 0$ and $a_j = 0,\ j > n.$

Let us define on G the relation
$f \leq_G g \Leftrightarrow f = g \ or\ \beta(g - f) > 0.$

Theorem: *The relation $\leq_G$ is a compatible total order on G.*

Proof.

We will begin by showing that $\leq_G$ is a partial order.

It is trivial to show that the relation is reflexive and symmetric. To prove transitivity, suppose that $f \leq_G g$ and $g \leq_G h$. We will study the different cases that can occur:

1) $f = g = h$

In this case transitivity is trivial.

2) $f = g \text{ and } \beta(h-g) > 0$

Since $f = g$ and $\beta(h-g) > 0$, then $\beta(h-f) > 0$ which implies that $f \leq_G h$, by the definition of $\leq_G$.

3) $\beta(g-f) > 0$ and $g = h$

The proof of this case is analogous to case 2.

4) $\beta(g-f) > 0$ and $\beta(h-g) > 0$

Let

$g - f = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$, with $a_n > 0$, and

$h - g = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m$, with $b_m > 0$.

Suppose that $n > m$. Then,

$h - f = (h-g) + (g-f) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_m + b_m)x^m + \ldots + a_nx^n$,

where $a_n > 0$ by hypothesis. Then $\beta(h-f) > 0$, which implies that $f \leq_G h$.

If, on the other hand, $n < m$, we have

$h - f = (h-g) + (g-f) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n + \ldots + b_mx^m$,

which implies that $\beta(h-f) > 0$, since $b_m > 0$. Then $f \leq_G h$.

If $n = m$, we have

$h - f = (h-g) + (g-f) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_m + b_m)x^m$.

Since $a_m, b_m > 0$, then $a_m + b_m > 0$, i.e., $\beta(h-f) > 0$, and then $f \leq_G h$.

So the relation is transitive.

Consequently $\leq_G$ is a partial order relation on G.

To show that is a total order suppose that $f >_G g$. Then $f \neq g$ and $\beta(g-f) < 0$. Therefore $\beta(f-g) > 0$, which implies that $g \leq_G f$ by the definition of $\leq_G$.

T show that $\leq_G$ is compatible, suppose that $f \leq_G g$, i.e., $f = g$ or $\beta(g-f) > 0$. We have to show that $f + h \leq_G g + h$.

If $f = g$ the proof is trivial since $f + h = g + h$.
If $\beta(g - f) > 0$ then $\beta[(g+h)-(f+h)] = \beta(g-f) > 0$, therefore we have that $\leq_G$ is a compatible total order on G. ■

Using the Bruck-Reilly extension $BR(G,\theta)$

Define on G the endomorphism θ, such that $\theta(f) = \dfrac{df}{dx}$ and consider the Bruck-Reilly extension $BR(G,\theta)$ such that the morphism $\phi : BR(G,\theta) \to Z$, defined by

$$\phi(b^r g a^s) = s - r$$

is isotone, using the usual order on integers.

In [1], we have that the relation

$$b^r g a^r \leq_N b^s h a^s \Leftrightarrow \begin{cases} \theta^{s-r}(g) \leq_G h & ,if\ s \geq r \\ g < \theta^{r-s}(h) & ,if\ s < r \end{cases},$$

defined on $N = Ker\phi = \{b^r g a^r, r \in IN^0\}$ is a compatible total order on N.
We also have in [1], the compatible total order on $BR(G,\theta)$ defined by:

$$b^r g a^s \leq_N b^m h a^n \Leftrightarrow s - r < n - m\ or$$

$$\begin{cases} s - r = n - m \quad and\ r > m\ and\ g <_G \theta^{r-m}(h) \\ \qquad\qquad\qquad or\ r \leq m\ and\ \theta^{m-r}(g) \leq_G h \end{cases}.$$

We will translate this relations to the set $P(x)$.
Let

$$g = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n,\ a_n \neq 0,$$

and

$$h = b_0 + b_1 x + b_2 x^2 + \ldots + b_m x^m,\ b_m \neq 0,$$

be two polynomials of degree n and m, respectively.
On the case $s \geq r$, we have

$$\theta^{s-r}(g) = \frac{d^{s-r}}{dx^{s-r}}(a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n)$$

$$= \begin{cases} 0 & ,if\ s - r > n \\ \displaystyle\sum_{j=0}^{n-(s-r)} \frac{(s-r+j)!}{j!} a_{s-r+j}.x^j & ,if\ s - r \leq n \end{cases}.$$

Using the compatible total order defined on G, we have:

$$\theta^{s-r}(g) \leq_G h \Leftrightarrow \begin{cases} \qquad \dfrac{d^{s-r}}{dx^{s-r}} g = h & \\ or \qquad b_m > 0 & ,if\ m > n-(s-r) \\ or \quad b_m > \dfrac{n!}{[n-(s-r)]!}.a_n & ,if\ m = n-(s-r) \\ or \qquad a_n < 0 & ,if\ m < n-(s-r) \end{cases}$$

Analogously, in case $s < r$:

$$\theta^{r-s}(h) = \frac{d^{r-s}}{dx^{r-s}}(b_0 + b_1 x + b_2 x^2 + \dots b_m x^m)$$

$$= \begin{cases} 0 & ,if\ r-s > m \\ \displaystyle\sum_{j=0}^{m-(r-s)} \frac{(r-s+j)!}{j!} b_{r-s+j}.x^j & ,if\ r-s \leq m \end{cases},$$

and using again the order on G, we have:

$$g \leq_G \theta^{r-s}(h) \Leftrightarrow \begin{cases} \qquad g = \dfrac{d^{r-s}}{dx^{r-s}} h & \\ or \qquad a_n < 0 & ,if\ n > m-(r-s) \\ or \quad \dfrac{m!}{[m-(r-s)]!}.b_m > a_m & ,if\ n = m-(r-s) \\ or \qquad b_m > 0 & ,if\ n < m-(r-s) \end{cases}.$$

The order on $N = Ker\phi$ is

$$b^r g a^r \leq_N b^s h a^s \Leftrightarrow \begin{cases} s \geq r & \begin{cases} \qquad \dfrac{d^{s-r}}{dx^{s-r}} g = h & \\ or \qquad b_m > 0 & ,if\ m > n-(s-r) \\ or \quad b_m > \dfrac{n!}{[n-(s-r)]!}.a_n & ,if\ m = n-(s-r) \\ or \qquad a_n < 0 & ,if\ m < n-(s-r) \end{cases} \\ s < r & \begin{cases} \qquad g = \dfrac{d^{r-s}}{dx^{r-s}} h & \\ or \qquad a_n < 0 & ,if\ n > m-(r-s) \\ or \quad \dfrac{m!}{[m-(r-s)]!}.b_m > a_m & ,if\ n = m-(r-s) \\ or \qquad b_m > 0 & ,if\ n < m-(r-s) \end{cases} \end{cases}$$

,

that we will call *(i)*.

Now we can define the compatible total order on Bruck-Reilly extension $BR(G,\theta)$. Therefore

$$b^r g a^s \leq b^u h a^v \Leftrightarrow \begin{cases} v-u>s-r \\ or \\ v-u=s-r \quad and \begin{cases} \theta^{u-r}(g) \leq_G h & ,if\, u \geq r \\ g <_G \theta^{r-u}(h) & ,if\, u<r \end{cases} \end{cases},$$

Which we can translate by:

$b^r g a^s \leq b^u h a^v \Leftrightarrow v-u>s-r$

or

$$v-u=s-r \text{ and } u \geq r \text{ and } \begin{cases} \dfrac{d^{u-r}}{dx^{u-r}} g = h & \\ or \quad b_m > 0 & ,if\, m>n-(u-r) \\ or \quad b_m > \dfrac{n!}{[n-(u-r)]!} a_n & ,if\, m=n-(u-r) \\ or \quad a_n < 0 & ,if\, m<n-(u-r) \end{cases}$$

or

$$v-u=s-r \text{ and } u<r \text{ and } \begin{cases} g = \dfrac{d^{r-u}}{dx^{r-u}} h & \\ or \quad a_n < 0 & ,if\, n>m-(r-u) \\ or \quad \dfrac{m!}{[m-(r-u)]!} b_m > a_n & ,if\, n=m-(r-u) \\ or \quad b_m > 0 & ,if\, n<m-(r-u) \end{cases}$$

,

where n and m are the degrees of g and h, respectively. We shall call this expressions *(ii)* and *(iii)*, respectively.

Example

For a better understanding of the order relation we shall consider the polynomials g and h:

$g=1+3x-x^2+x^3+x^4$ and $h=1+2x-x^2+5x^3$,

with degrees $n=4$ and $m=3$, respectively.

Using the compatible total order defined on G, we have that $h \leq_G g$, since $\beta(g-h)=1-0>0$.

Consider the elements of $N=\ker\phi$, $b^3 g a^3$ and $b^5 h a^5$. We have $b^3 g a^3 \leq_N b^5 h a^5$ because, using *(i)* defined on *N*, $s=5>r=3$ and $b_3=5>0$.

Let us consider now two elements of $BR(G,\theta)$, $b^3 g a^2$ and $b^4 h a^3$.

It is easy to show that $b^3ga^2 \leq b^4ha^3$, since $v-u=3-4=s-r=2-3$ and $u>r$. Therefore, using *(ii)*, $m=3=n-(u-r)=4-(4-3)$ and $b_3=5>\frac{4!}{[4-(4-3)]!}.a_4=4.1$.

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