TIME AS A COORDINATE VARIABLE

Hassan R. Althubyani

Khaled I. Nawafleh

Department of Physics, Mu'tah University, Al-Karak, Jordan

Abstract

We have investigated the equations of motion for parametrized invariant Lagrangian systems. We applied the technique of separation of variables and the method of canonical transformations to solve the Hamilton-Jacobi equation. The quantization of parametrized invariant Lagrangian systems is investigated using the WKB approximation.

Keywords: Parametrized Lagrangians, Hamilton-Jacobi, Quantization PACS numbers: 11.10.Ef

Introduction

Time plays a central and peculiar role in quantum mechanics. In the standard nonrelativistic quantum mechanics, one can describe the motion of a system by using the canonical variables which are functions of time only. Time is the sole observable assumed to have a direct physical significance; but it is not a dynamical variable itself. It is an absolute parameter treated differently from the other coordinates, which turn out to be operators in quantum mechanics.

It is well known that any standard Hamiltonian system can be transformed to a constrained system with a vanishing Hamiltonian by going to an arbitrary reparametrization of time, thereby introducing the original time coordinate as a new dynamical variable.

Parametrization invariance is a way that takes the time as an extra canonical variable of the system on the same footing as the position variable, and it is then easy to introduce a non-canonical structure in the extended phase-space by including an invariant parameter through the action integral which will play the role of the time. Hence, the canonical transformation here is implemented in an extended phase space, where the time and its conjugate momentum are included [1-5].

The usual way to study the parametrization invariance of a system is by using the Dirac method of canonical analysis [4,5]. Because not all the momenta are independent due to the invariance under parametrizations, this approach requires that a constraint on the system be introduced. For a parametrized particle, this constraint is at the classical level the Hamilton-Jacobi equation (HJE) and at the quantum level the Schrodinger equation. So the Dirac method associates to the symmetry of parametrizations the classical or quantum evolution equations.

Another powerful approach to study parametrization invariance of a system is by using the Hamilton-Jacobi (HJ) formalism for constrained systems [6-10], based on Caratheodory's equivalent Lagrangians method [11]. This formalism does not differentiate between the first and second class constraints as in Dirac's method, we do not need any gauge fixing terms and the action provided by HJ is useful for the path integral quantization method of the constrained systems.

In this work, we want to generalize the above mentioned procedure to obtain the Hamilton-Jacobi equations for the reparametrized invariant Lagrangian systems and make use of its singularity to write the equations of motion as total differential equations in many variables. The interesting point of the procedure is that on the one hand we get the classical and quantum evolution equations for the reparametrized invariant systems and on the other hand we also obtain a classical action that can be quantized using the WKB approximation [12-15]. Another interesting property of the method is that it can be naturally extended to field theory.

Basic Tools

This section is concerned with the theoretical framework for the HJ formalism and the WKB approximation of reparametrized Lagrangian systems.

Hamilton–Jacobi Equation

The fundamental functional in the Lagrangian formalism is the action

$$S = \int_{t_0}^{t_1} L(q, \dot{q}) dt \,. \qquad (2.1)$$

The requirement that S be minimized implies that the variation of S vanishes

$$\delta S = \delta \int_{t_0}^{t_1} L(q, \dot{q}) dt = 0 \,.$$

This variation leads to

$$\delta S = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = 0 \,.$$

Integrating the second term by parts, we obtain

$$\delta s = \int_{t_0}^{t_1} \left[\frac{\partial L}{\partial \dot{q}} \delta q \right] + \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q dt = 0 \quad (2.2)$$

The second term vanishes because the motion satisfies Lagrange's equations. In the first term, we set $\delta q(t_0) = 0$ and replace $\delta q(t_1)$ with δq because t_1 adopts any value of t greater than t_0.

Invoking the definition of momenta:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

we arrive at the equivalence $\delta S = p \delta q$.

For an n dimensional system, this has the form

$$\delta S = p_i \delta q^i \quad i = 1, 2, \ldots, n. \qquad (2.3)$$

Therefore a relationship between the action and momenta follows directly

$$p_i = \frac{\partial S}{\partial q_i}. \qquad (2.4)$$

Another necessary result is produced by examining the total time derivative of the action. Directly from the definition of the action (2.1), we observe that

$$\frac{dS}{dt} = L \,. \qquad (2.5)$$

However, by viewing the action as a function of only coordinates and time, it is obvious, using (2.4), that

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_i} \dot{q}_i = \frac{\partial S}{\partial t} + p_i \dot{q}_i \,. \qquad (2.6)$$

Comparing (2.5) and (2.6), we obtain

$$\frac{\partial S}{\partial t} + H(p, q, t) = 0 \qquad (2.7)$$

where the Hamilton's function *H* is the Legendre transform of the Lagrangian with respect to the variable $\dot{q}$

$$H(p, q) = p\dot{q} - L(q, \dot{q}, t) \,.$$

The momenta in eq.(2.7) can be replaced using eq.(2.4) to produce the first order partial differential equation called the HJE [16,17]

$$\frac{\partial S}{\partial t}+H(q_1,....,q_n,\frac{\partial S}{\partial q_1},....,\frac{\partial S}{\partial q_n},t)=0 . \qquad (2.8)$$

We now establish the connection between the complete integral of the (HJE) and solutions of Hamilton's equations. We use the function W(q,t;α) as the generating function for a canonical transformation from the original coordinates (p,q) to (α,β); therefore, our new position coordinates are ($\beta_1,\beta_2,\ldots,\beta_n$) and new momenta coordinates are ($\alpha_1,\alpha_2,\ldots,\alpha_n$). With this generating function, the new Hamiltonian vanishes everywhere; thus, the transformed Hamilton's equations become

$$\dot{\alpha}_i=0 \ , \ \dot{\beta}^j=0$$

and we solve for the position coordinates q as functions of t, α and β using the relationships

$$\beta^i=\frac{\partial W}{\partial \alpha_i} .$$

An important technique for the determination of complete integral for the HJE of the system is the method of separation of variables. Under certain conditions it is possible to separate the variables in the HJE, the solution can be then always reduced to quadrature. In practice, the Hamilton-Jacobi technique becomes a useful computational tool only when such a separation can be effected. HJE plays a good role and becomes beautiful treatment when it can be solved using separation of variables, which directly identifies constant of motion.

In general, a coordinate q_i is said to be separable in the HJE when Hamilton's principal function S can be split into two additive parts, one of which depends only on the coordinate q_i ; whereas the second is independent of q_i , which means time dependent part.

In the cases to which we shall apply the method of separation of variables, the Hamiltonian will be time independent. Therefore the HJ equation (2.8) for this system will be in the following form:

$$\frac{\partial S(q,t,\alpha)}{\partial t}+H\ (q,\frac{\partial S}{\partial q})=0 \qquad (2.9)$$

We can separate the variables as

$$S(q,\alpha,t) = W(q,\alpha) + f(t)\,, \tag{2.10}$$

where the time-independent function W(q,α) is sometimes called Hamilton's characteristic function.

Differentiate Eq (2.10) with respect to time, we find

$$\frac{\partial S}{\partial t} = \frac{\partial f}{\partial t}\,,$$

and using the Hamilton-Jacobi equation (2.9), then we have

$$\frac{\partial f}{\partial t} = -H\,. \tag{2.11}$$

The left hand sides of equation (2.11) depends on t, whereas the right hand side depends on q, so that each side equal to a constant independent of both q and t, the time t can be separated if the Hamiltonian does not depend on time explicitly. In that case, the time derivative $\frac{\partial S}{\partial t}$ in the HJE must be a constant, usually denoted by (- α), giving the separated solution. Then we obtain

$$S(q,\alpha,t) = W(q,\alpha) - \alpha\, t\,. \tag{2.12}$$

Hamilton-Jacobi Treatment of Reparametrized Systems

In the cases of nonrelativistic and relativistic point-particle mechanics, generally covariant systems may be obtained by promoting *t* to a dynamical variable [18-20]. The idea behind this transformation is to treat symmetrically the time and the dynamical variables of the system. This is achieved by taking *t* as a function of an arbitrary parameter τ : $t=t(\tau), q=q(\tau)$ (e.g., τ is the "proper time" in relativity theory). The arbitrariness of the label time τ is reflected in the invariance of the action under the time reparametrization. If S is the action integral, then

$$S = \int L(q,\frac{dq}{dt})dt \equiv S^{*} = \int L^{*}(q,\frac{dq}{d\tau})\,d\tau\,. \tag{2.13}$$

Thus, we can express the action integral with respect to τ in the same form as with respect to t. This shows that the equations of motion which follow from the action principle must be invariant under the transformation from t to τ. The equations of motion do not refer to any absolute time. We have, therefore, a special form of Hamiltonian theory; but this form is not really so special because, starting with any Hamiltonian, it is always permissible to take the time variable as an extra coordinate and bring the theory into a form in which the Hamiltonian is equal to zero. The general rule for doing this is the following: we take t and put it equal to another dynamical coordinate q_0. We set up a new Lagrangian:

$$L^* = L^*\left(q_0, q_i, \frac{dq_0}{d\tau}, \frac{dq_i}{d\tau}\right), \quad i = 1, 2, \ldots, N \qquad (2.14)$$

L^* involves one more degree of freedom than the original L.

L^* is not equal to L; but

$$L\,dt = L^* d\tau\,. \qquad (2.15)$$

Thus, the action is the same whether it refers to L^* and τ or to L and t.

This special case of the Hamiltonian formalism, where the Hamiltonian

$$H_0{}^* = p_0 \frac{dq_0}{d\tau} + p_i \frac{dq_i}{d\tau} - L^* \equiv 0 \qquad (2.16)$$

is what is needed for a relativistic theory, because in such a theory we do not want to have one particular time playing a special role. Instead, we want to have the possibility of various times τ which are all on the same footing.

However, for reparametrized systems, the HJE takes the form

$$H_t' = p_t + H_t = 0\,, \qquad (2.17)$$

p_t being the generalized momentum associated with t and H_t of the originally noncovariant formulation.

WKB Approximation of Reparametrized Lagrangians

It is well known that the HJE for dynamical systems leads naturally to a semiclassical approximation; namely, WKB. The Schrödinger equation in one dimension for a single particle in a potential $V(q)$ reads

$$i\hbar \frac{\partial \psi(q,t)}{\partial t} = \left[-\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} + V(q)\right]\psi(q,t)\,. \qquad (2.18)$$

Writing $\psi(q,t) = EXP\left(\frac{iS(q,t)}{\hbar}\right)$ and considering the expansion

$$S(q,t) = S_0 + \hbar S_1 + \hbar^2 S_2 + ...$$

such an expansion is used in the WKB approximation where S is real to leading order; at this point we have not said anything about the reality of S, so the above equation is just a mathematical identity. Then we have

$$-\frac{\partial S}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 S}{\partial q^2} + \left(\frac{\partial S}{\partial q}\right)^2 + V(q). \qquad (2.19)$$

If we assume that $\hbar \to 0$, which is the 'classical limit' in quantum mechanics, then we see that

$$-\frac{\partial S}{\partial t} = \left(\frac{\partial S}{\partial q}\right)^2 + V(q).$$

More general as

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0 \qquad (2.20)$$

which is just the HJE. Thus we see that in the classical limit, $\hbar \to 0$, the Schrodinger equation is just the HJE when the dynamical coordinates and momenta are turned into their corresponding operators :

$$q_i \to q_i \qquad (2.21)$$

$$p_i \to \hat{p}_i = \frac{\hbar}{i}\frac{\partial}{\partial q_i}$$

$$p_o \to \hat{p}_o = \frac{\hbar}{i}\frac{\partial}{\partial t}$$

In the classical limit $\hbar \to 0$, the condition (2.17) implies that

$$\hat{H}'_t \psi = o. \qquad (2.22)$$

Examples

In this section, we illustrate the canonical methods for treating some particular cases of reparametrized Lagrangian systems, the application of the methods is straightforward. Of course the canonical methods provide the description of motion in phase space, which is the basis for further insights and generalizations.

Nonrelativistic Reparametrized Particle System

We start by considering a nonrelativistic particle moving in one-dimensional space with dynamical variables x and with t denoting the ordinary physical time parameter. The action for this simplest model may be written as

$$S = \int L dt = \int\left(\frac{1}{2}m\dot{x}^2 - V(x)\right)dt , \qquad (3.1)$$

where m is the mass of the particle, $\dot{x} = dx/dt$ is its velocity and $V(x)$ is the potential.

In the action (3.1) t is an absolute parameter, taking t to be a function of local time τ, t = t(τ), then eq.(3.1) reduces to

$$S^* = \int L^* d\tau = \int\left(\frac{1}{2}m\frac{\dot{x}^2}{\dot{t}} - V(x)\dot{t}\right)d\tau ,$$

Where

$$L^* = L\dot{t} = \frac{1}{2}m\frac{\dot{x}^2}{\dot{t}} - V(x)\dot{t} , \qquad (3.2)$$

the dot now standing for $\frac{d}{d\tau}$.

The generalized momenta corresponding to L^* are

$$p_x = \frac{\partial L^*}{\partial \dot{x}} = \frac{m\dot{x}}{\dot{t}} , \qquad (3.3)$$

$$p_t = \frac{\partial L^*}{\partial \dot{t}} = -\frac{1}{2}m\frac{\dot{x}^2}{\dot{t}^2} - V(x). \qquad (3.4)$$

Substituting the value of $\dot{x}$ from eq.(3.3) into equation (3.4), then we have

$$p_t = -\frac{p_x^2}{2m} - V(x) \equiv -H_t .$$

This equation can be written as:

$$H_t' = p_t + H_t = 0 , \qquad (3.5)$$

Where

$$H_t = \frac{p_x^2}{2m} + V(x).$$

It is easy to show that the canonical Hamiltonian H_0^* is identically zero

$$H_0^* = p_t\dot{t} + p_x\dot{x} - L^* = 0 .$$

The corresponding HJE of eq.(3.5), is

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2+V(x)=0\,. \tag{3.6}$$

Using the separation of variables technique we can write

$$S(x,\alpha,t)=W(\alpha,x)-\alpha t\,. \tag{3.7}$$

Then eq.(3.6) becomes

$$-\alpha+\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2+V(x)=0\,,$$

which can be solved for W(x,α) as

$$W(x,\alpha)=\int\sqrt{2m(\alpha-V)}dx\,.$$

Therefore the general solution of S takes the form

$$S(x,\alpha,t)=\int\sqrt{2m(\alpha-V)}dx-\alpha\,t\,. \tag{3.8}$$

The equations of motion for x and p_x are

$$\beta=\frac{\partial S}{\partial\alpha}=-t+\int\frac{m}{\sqrt{2m(\alpha-V)}}dx\,, \tag{3.9}$$

$$p_x=\frac{\partial S}{\partial x}=\sqrt{2m(\alpha-V)}\,. \tag{3.10}$$

Equations (3.9) and (3.10) can be sloved for x and p_x in terms of t and α.

Applying now the operator H_t' in eq.(3.5) as an operator on the wave equation $\psi(x,t)$, we get

$$H_t'\psi=\left[\frac{\hbar}{i}\frac{\partial}{\partial t}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right]\psi\,. \tag{3.11}$$

With $\psi(x,t)=\exp(\frac{iS}{\hbar})$, and S is given in Eq.(3.8), then we have

$$H_t'\psi=\left[-\alpha-\frac{\hbar^2}{2m}\left(\frac{-2m}{\hbar^2}(\alpha-V)-\frac{im}{\hbar\sqrt{2m(\alpha-V)}}\frac{\partial V}{\partial x}\right)+V\right]\psi\,.$$

After simplifications we obtain

$$H_t'\psi=\left[-\alpha+\alpha-V+\frac{i\hbar}{2\sqrt{2m(\alpha-V)}}\frac{\partial V}{\partial x}+V\right]\psi\,. \tag{3.12}$$

In the semiclassical limit $\hbar \to 0$, Eq.(3.12) is identically zero, that is

$$H_t^{'}\psi \equiv 0. \tag{3.13}$$

So the corresponding quantum-mechanical operator annihilates the wave function, which is precisely the Schrödinger equation.

Relativistic Reparametrized Particle System

Using the physical coordinates $x(t)$, the relativistic particle action reads

$$S = \int L dt = -mc\int \sqrt{c^2 - \left(\frac{dx}{dt}\right)^2}\, dt\,, \tag{3.14}$$

where m is the mass of the particle, and c is the speed of the light. Introducing an arbitrary paramertrization $x(\tau)$, $t(\tau)$ of the trajectory, the action requires the reparametrization invariant form

$$S^* = \int L^* d\tau = -mc\int \dot{t}\sqrt{c^2 - \frac{\dot{x}^2}{\dot{t}^2}}\, d\tau\,, \tag{3.15}$$

Or

$$S^* = \int L^* d\tau = -mc\int \sqrt{c^2\dot{t}^2 - \dot{x}^2}\, d\tau\,, \tag{3.16}$$

where

$$L^* = -mc\int \sqrt{c^2\dot{t}^2 - \dot{x}^2}\,. \tag{3.17}$$

We introduce the generalized canonical momenta as

$$p_x = \frac{\partial L^*}{\partial \dot{x}} = \frac{mc\dot{x}}{\sqrt{c^2\dot{t}^2 - \dot{x}^2}}\,, \tag{3.18}$$

$$p_t = \frac{\partial L^*}{\partial \dot{t}} = \frac{-mc^3\dot{t}}{\sqrt{c^2\dot{t}^2 - \dot{x}^2}} \equiv -H_t\,. \tag{3.19}$$

In fact eq.(3.19) may be written as

$$H_t^{'} = p_t + H_t = 0\,, \tag{3.20}$$

and eq.(3.18) can be solved for $\dot{x}$

$$\dot{x} = \frac{c\dot{t}p_x}{\sqrt{p_x^2 + m^2c^2}}\,. \tag{3.21}$$

Then eq.(3.20) reduces to

$$H_t' = p_t + c\sqrt{p_x^2 + m^2c^2} = 0\,. \tag{3.22}$$

It is obvious to note that H_0^* is identically zero:

$$H_0^* = p_t\dot{t} + p_x\dot{x} - L^* \equiv 0 . \qquad (3.23)$$

In details we have

$$H_o^* = \frac{-mc(c^2\dot{t}^2 - \dot{x}^2)}{\sqrt{c^2\dot{t}^2 - \dot{x}^2}} + \frac{mc(c^2\dot{t}^2 - \dot{x}^2)}{\sqrt{c^2\dot{t}^2 - \dot{x}^2}} = 0. \qquad (3.24)$$

The corresponding HJE for Eq.(3.22) is

$$\frac{\partial S}{\partial t} + c\sqrt{\left(\frac{\partial S}{\partial x}\right)^2 + m^2c^2} = 0 . \qquad (3.25)$$

Making use the separation of variables techniques we can write

$$S(x,\alpha,t) = W(\alpha,x) - \alpha t . \qquad (3.26)$$

With the aim of Eq.(3.25), Eq.(3.26) can be solved for W(x,α) as

$$W(x,\alpha) = \int\sqrt{\left(\frac{\alpha^2}{c^2} - m^2c^2\right)}dx . \qquad (3.27)$$

It follows that

$$S = \int\sqrt{\left(\frac{\alpha^2}{c^2} - m^2c^2\right)}dx - \alpha t . \qquad (3.28)$$

Now the equations of motion for x and p_x can be obtained by using the canonical transformations as

$$\beta = \frac{\partial S}{\partial \alpha} = -t + \int \frac{\alpha}{c^2\sqrt{\frac{\alpha^2}{c^2} - m^2c^2}}dx , \qquad (3.29)$$

And

$$p_x = \frac{\partial S}{\partial x} = \sqrt{\left(\frac{\alpha^2}{c^2} - m^2c^2\right)} . \qquad (3.30)$$

Following the previous procedure for the quantization of the system. With

$$\psi(x,t) = \exp\left(\frac{i}{\hbar}\left[-\alpha t + \int\sqrt{\left(\frac{\alpha^2}{c^2} - m^2c^2\right)}dx\right]\right). \qquad (3.31)$$

The result of the operation $H_t'\psi$ reads

$$H_t'\psi(x,t) = \left[\frac{\hbar}{i}\frac{\partial}{\partial t} + c\sqrt{-\hbar^2\frac{\partial^2}{\partial x^2} + m^2c^2}\right]\psi(x,t).$$

Then we have

$$H_t'\psi(x,t) = \left[-\alpha + c\sqrt{-\hbar^2\frac{\partial^2}{\partial x^2} + m^2c^2}\right]\psi(x,t). \tag{3.32}$$

In the semiclassical limit $\hbar \to 0$, this implies that

$$H_t'\psi(x,t) \equiv 0 \tag{3.33}$$

which is just the conventional time-dependent Schrödinger equation as required.

Conclusion

The Hamilton-Jcobi partial differential equations for reparametrized Lagrangian systems are discussed using the canonical method. It has been shown that any standard Hamiltonian system can be transformed into a constrained system with vanishing Hamiltonian by going to an arbitrary reparametrization of time. In doing so, the time variable is treated on the same level as the other dynamical variables. Thus, we have an extended phase space that includes a new coordinate, the time, whose conjugate momentum represents the total energy of the system.

Due to the reparametrization invariance, the quantity H_t' vanishes for any solution, $H_t' = p_t + H_t = 0$. So the corresponding quantum-mechanical operator annihilates the wave function $H_t'\psi = 0$, which is precisely the Schrödinger equation, $i\hbar\frac{\partial \psi}{\partial t} = H_t'\psi$.

Further, the Hamilton-Jacobi function S is determined in configuration space in the same manner as for regular systems. Finding S enables us to get the solutions of the equations of motion. These solutions are obtained in terms of the time and the coordinates that correspond to dependent momenta.

The success of this work has been demonstrated for two applications. The first is an illustrative example in one-dimensional dynamics that describes the concept of nonrelativistic parametrized dynamics. It has been shown that the quantization procedure applied to the initial mechanical system, after promoting the time to become a dynamical variable, yields the correct equation for the wave function, which is just the conventional time-dependent

Schrödinger equation. The second application is quantization of the motion of a relativistic parametrized particle system.

References:

Gitman D.M., and Tyutin I.V. *Quantization of fields with constraints*. Springer-Verlag, Berlin, Heidelberg (1990).

Fülop G., Gitman D. M. and Tyutin I.V. *International Journal of Theoretical Physics* 38 (1999) 1941.

Rosales J. J., Tkach V.I. and Pashnev A.I., *Physics Letters A* 286 (2001) 15.

Dirac P. A. M. *Canadian Journal of Mathematics,* 2 (1950) 129.

Dirac P. A. M. *Lectures on Quantum Mechanics*. Belfer Graduate School of Science, Yeshiva University, New York (1964).

Güler Y**.** *Nuovo Cimento B*, 107 (1992) 1143.

Muslih S. I. and Güler Y., *Nuovo Cimento B*, 113 (1998) 277.

Baleanu D., *Nuovo Cimento B*, 119 (1) (2004) 89.

Pimentel B.M., Teiteira R.G. and Tomazelli J.L. *Annals of Physics* 267 (1998) 75.

Baleanu D. and Güler Y. *Nuovo Cimento B*, 1158, 6 (2003) 615.

Caratheodory C. *Calculus of Variations and Partial Differential Equations of the First Order Part II*, Holden-Day San Francisco, Calif, USA (1967).

Rabei E.M., Nawafleh K.I., and Ghassib H.B., *Physical Review. A* 66, 024101(2002)1.

Rabei E.M., Nawafleh K.I., and Ghassib H.B., *Journal of Dynamical Systems and Geometric Theories*, 2 (2004) 1.

Nawafleh K.I., Rabei E.M., and Ghassib H.B., *International Journal of Modern Physics A*. 19 (3) (2004) 347.

Nawafleh K.I., Rabei E.M., and Ghassib H.B., *Turkish Journal of Physics*. 29 (2005) 151.

Goldstein, H. *Classical Mechanics*, 2nd edition, Addison-Wesley, Reading-Mass (1980).

Arnold, I.V. *Mathematical Methods of Classical Mechanics*, 2nd edition, Springer-Verlag, Berlin (1989).

Henneaux, M. and Teitelboim,C. *Quantization of Gauge Systems*. Princeton Univ. Press (1992).

Isham, C. and Kuchar, K. *Annals of Physics* 164 (1985) 316.

Lucio, J., Antonio, J. and David J. *Physics Letters A* 219 (1996) 150.