BIRKHOFF INTEGRAL IN QUASY-BANACH SPACES

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Abstract

Aim of Investigation: Birkhoff integral of functions taking values in a Banach space plays an important role in the modern theory of integration (see [1],[4]). Therefore, it is with great interest the study of the existence of Birkhoff integral for the class of functions with values in a quasy – Banach space, and the properties of the Birkoff integral's in this case. The other aim of this paper is analysis of validity of some of the known theorems for integration theory, at of Birkoff integral's case.

Conclusions: The extension of the meaning of the Birkhoff integral, for functions with values in quasy – Banach space, together with the meanings of Aumann and Bochner integrals, allows us to see the class of functions with values in quasy -Banach spaces with a lot of interest in terms of integration.

Keywords: Birkhoff integral, quasy-Banach space, norm convergence

Introduction

Recalling the construction of Fréchet integral, consider the function f : $\sum \to R$, where $\sum$ is an σ-algebra of measurable sets and R is the real number system. Let be $A \in \sum$ and Δ a partition of A composed of finite or countable amount sets A_i (the measures of them are $\mu(A_i)$). We build on this partition amount of upper and lower integral respectively as follows:

$$I^*(f,\Delta) = \sum_i \mu(A_i) \sup_{x \in A_i} f(x) \quad \text{and} \quad I_*(f,\Delta) = \sum_i \mu(A_i) \inf_{x \in A_i} f(x)$$

assuming that both series converge unconditionally. It is clear that, for every two partitions Δ and Δ' is true inequality $I_*(f,\Delta) \le I^*(f,\Delta)$.Therefore there is a number x such that $I_*(f,\Delta) \le x \le I^*(f,\Delta)$ and, if this number is only then f(x) is Fréchet integrable and number x is integral of function f. Birkhoff integral obtained from Fréchet integral making two changes: The system of real numbers R is replaced with a quasy-Banach space Ɓ, and sets of the upper and lower integrals are defined respectively as the smallest closed and convex set containing sums $\sum_i \mu(A_i) f(x_i)$ where $x_i \in A_i$, assuming again the unconditional convergence of all series.

Terminology and preliminaries

If Ɓ is a vector space and θ is its origin, then we have properties:

a) $\forall B_1, B_2 \subset$ Ɓ, $B_1+B_2 = B_2+B_1$

b) $\forall B_1, B_2, B_3 \subset$ Ɓ, $B_1+(B_2+B_3) = (B_1+B_2)+B_3$

c) $\forall b \in R$ and $\forall$ B1,B2⊂Ɓ, b(B1+B2) = bB1+bB2

d) $\forall$ b1,b2∈R and ∀B⊂Ɓ, b1(b2B) = b1b2B

e) ∀ B⊂Ɓ, 1·B = B

f) ∀ B⊂Ɓ ,B +θ = B and 0·B = 0.

Therefore, if we define as the space 'Vectoroid' the system that satisfies the conditions a)-f) can state that: All non empty subset of Ɓ are elements of a 'Vectoroid' space.

- Easily seen that: Every convex subset B⊂Ɓ is convex hull of themselves.
- Admit that: The subset B⊂Ɓ is convex if and only if $(m_1+m_2)B = m_1B+m_2B$ for every $m_1, m_2 \geq 0$.

Proof

Let be B a convex set. Take an element $m_1x + m_2y$ where x, y∈B.

Since B is a convex set then $m_1x + m_2y = (m_1+m_2)\left(\frac{m_1}{m_1+m_2}x + \frac{m_2}{m_1+m_2}y\right) = (m_1+m_2)z$

where $z = \frac{m_1}{m_1+m_2}x + \frac{m_2}{m_1+m_2}y \in B$. Thus $m_1B+m_2B \subset (m_1+m_2)B$.

The other inclusion is immediate from the properties of the scalar multiplication of vectors.

Conversely, if we take x, y∈B and $0 \leq m_1 \leq 1$, $m_2=1-m_1$ then $m_1x+m_2y \in m_1B+m_2B=(m_1+m_2)B= B$.

- Based on d property, we can assert that:

Theorem 1

For every subset B∈Ɓ, Co(mB) = mCo(B) and Co(A+B) = Co(A) + Co(B).

So, the correspondence B→Co(B) is homeomorphism.

The Results

Quasy-norm and diameter of convex hull

Now onwards, Ɓ space is quasy-normed. So, every bounded subset B of Ɓ, we can put them into correspondence the size $\|B\|=\sup\{\|\beta\|: \beta\in B\}$. It's clear that:

Proposition 2

Correspondence $B\to\|B\|$ defined above is a quasy-norm.

Well we have specified the quasy-norm of an subset of Ɓ.

- We also B associate to a 'diameter' with $\rho(B)=\|B-B\|\leq 2K\|B\|$.

Observed that $\|B\|\leq\|Co(B)\|$ and $\rho(B)\leq 2K\rho(Co(B))$.

Limits, closure and unconditional summation of elements

Let be B a quasy-Banach space. It is known that (see [4]),Ɓ space can be seen like a topological vector space and so, it makes sense to talk about closing of sets on it.

Unlike the case of normed space, here we can only guarantee that $\left\|\sum_{i=1}^{r} B_i\right\| \leq \left\|\sum_{i=1}^{r} \overline{Co(B_i)}\right\|$, where B_i are arbitrary subset of Ɓ. In a particular case, when Bi are closed and convex sets, for example in case of set with an element, we can write $\left\|\sum_{i=1}^{r} B_i\right\| = \left\|\sum_{i=1}^{r} \overline{Co(B_i)}\right\|$.

Definition 3

A countable set $Z\subset$Ɓ with elements $\xi_1, \xi_2,\ldots$,(which need not be distinct) is called unconditionally summable to ξ if and only if every arrangement α of all the elements of Z gives a series $Z^{(\alpha)}:\xi_{\alpha(1)}+ \xi_{\alpha(2)}+\ldots$ convergent to ξ.The series $Z^{(\alpha)}$ are unconditionally convergent to ξ, under these conditions.

- The unconditionally convergent series of Ɓ are the elements of a vector space L.

Let B(Z) denote the set of the partial sums of the elements of Z. By the quasy-norm $\|Z\|$ of Z we mean $\|B(Z)\|$.Since $B(Z+Z')\subset B(Z)+B(Z')$ and $B(cZ) =cB(Z)$,we see that the L space is quasy-normed.

Now, we shall prove that L space is complete by this quasy-norm.

Theorem 4

The unconditionally convergent series of Ɓ are the elements of a second quasy-Banach space.

Proof

Let $Z_1, Z_2, Z_3, \ldots$ be any sequence of unconditionally convergent series of elements of Ɓ, such that to any ε>0 corresponds N as large as, that m, n>N imply $\|Z_m - Z_n\| < \frac{\varepsilon}{K}$ (a Cauchy sequence on L space).

It is clear that terms ξ_i^k of the Z_k are uniformly convergent Cauchy sequences, with limits ξ_i.

Let Z denote the formal series $\xi_1 + \xi_2 + \xi_3 + \ldots$.The proof is complete if Z is unconditionally convergent and $\lim_{n\to\infty} \|Z - Z_n\| = 0$.But to any ε>0 corresponds N as large as, that if m, n>N, then $\|Z_m - Z_n\| < \frac{\varepsilon}{K}$. So, we can find M as large as, that if M < k(l) <… < k(r), then $\left\| \sum_{i=1}^{r} \xi_{k(i)}^{N} \right\| < \frac{\varepsilon}{K}$ (the latter is provided by Z_n series convergence).

Follows that, under the same hypotheses,

$$\left\| \sum_{i=1}^{r} \xi_{k(i)} \right\| = \left\| \lim_{n\to\infty} \sum_{i=1}^{r} \xi_{k(i)}^{n} \right\| \le K \left\| \sum_{i=1}^{r} \xi_{k(i)}^{N} \right\| + K \frac{\varepsilon}{K} < 2\varepsilon .$$

So that, E must be unconditionally convergent.

Remember that in [4] is shown that: if x_n sequence tends to x then $\lim_{n\to\infty}\|x_n\| \le K\|x\|$.

But now, if we take any j(1)<j(2)<…<j(s),then for n>N,

$$\lim_{n\to\infty} \|Z - Z_n\| = \lim_{n\to\infty} \left\| \sum_{i=1}^{s} \xi_{j(i)} - \xi_{j(i)}^{n} \right\| \le K \left\| \sum_{i=1}^{s} \xi_{j(i)} \right\| < 2K\varepsilon = \varepsilon' .$$

Hence, the proof is completing.

Unconditional summation of subset on Ɓ

Suppose similarly θ is an aggregate of countable subsets $B_1, B_2, \ldots$ of Ɓ.θ will be called unconditionally summable to a given subset B if and only if every series $\beta_1 + \beta_2 + \beta_3 + \ldots$ ($\beta_i \in B_i$) is unconditionally convergent, and B is the locus of the sums of such series. We shall abbreviate this by writing $\sum_i B_i = B$.

In order that θ be unconditionally summable it is necessary as well as sufficient that to any ε>0 correspond N so large that N<k(1)<k(2)<…<k(r) implies $\|B_{k(1)} + \ldots + B_{k(r)}\| < \varepsilon$.For otherwise we could form an infinite series of elements from a sequence of such sets of subsets which was not unconditionally convergent no matter how the gaps between the different terms were filled in by elements from the remaining of θ.

Same as in the case of normed space (see [2]) shown that:

Theorem 5

If $\sum_i B_i = B$ and the aggregate $\overline{Co(B_i)}$ is unconditionally summable then $\sum_i \overline{Co(B_i)} \subset \overline{Co(B)}$ and $\overline{Co(B)} = \overline{\sum_i Co(B_i)} = \overline{\sum_i \overline{Co(B_i)}}$.

Admissible domains and Completely additive set functions.

We shall define as an admissible domain any σ -algebra Σ of measurable sets.

The integration will be defined relative to the σ-algebra Σ. It is natural that we should define a (single-valued) "set function" as a function F assigning to each set σ of Σ a single "value" F(σ) in Ɓ.

F is called completely additive if and only if the hypothesis that σ is the sum of finite or countable disjoint sets σ_i of Σ implies the conclusion that the values $F(\sigma_i)$ are unconditionally summable to F(σ).

Lemma 6

If F is completely additive, then the set of the F(σ) ($\sigma \in \Sigma$) has a finite upper bound.

Otherwise we could choose $\sigma_1, \sigma_2, \sigma_3, ...$ by induction so as to satisfy $||F(\sigma_1)|| > 1$, $||F(\sigma_{i+1})|| \geq 3||F(\sigma_i)||$ and the series of the $F(\sigma_i - \sigma_i(\sigma_1 + ... + \sigma_{i-1}))$ could not be unconditionally summable.

The (finite) least upper bound to $||F(\sigma_i)||$ will be called the quasy-norm of F denoted by ||F||.

Theorem 7

The completely additive set functions of ϑ to Ɓ are a quasy-Banach space Ƒ(σ,Ɓ).

Every property of Banach space is obvious except completeness. But since, the F_n (σ) are a uniformly convergent Cauchy sequence, it is obvious that they tend uniformly to a limit set function F.

It remains to prove that F is completely additive. But for each choice of $\sigma = \sigma_1 + \sigma_2 + \sigma_{3...}$,this is a corollary of Theorem 4.This completes the proof of Theorem 7.

Admissible point functions.

By a "function" (more precisely, point function) T of an admissible domain ϑ to a quasy-Banach space Ɓ we shall mean from now on a rule assigning to each point p of ϑ one or more "images" in Ɓ. More generally, if σ is any subset in ϑ, we shall use T(σ) to denote the subset of the images of the points of σ.

This defines the admissible functions as elements of a "vectoroid" space, which becomes a vector space if we restrict ourselves to single-valued functions.

Definition 8

A function T is called "summable" under the decomposition Δ of ϑ if and only if each $T(\sigma_i)$ is bounded, and the aggregate of the $\mu(\sigma_i)\ T(\sigma_i)$ is unconditionally summable.

Definition 9

If T is summable under Δ, then the set

$$I_{\Delta}(T) \equiv \overline{Co\left(\sum_i \mu(\sigma_i)T(\sigma_i)\right)}$$

is called the "integral range" of T relative to Δ.

Let be two partitions Δ and Δ_1 and $\Delta\Delta_1$ its cutting partition. Consider those functions T which are summable under the cutting partition. (in the case when the function T is a single-valued function enough that to be summable under the decomposition Δ and Δ_1).

Observed that:

$$I_{\Delta\Delta_1}(T) \subset I_{\Delta}(T) I_{\Delta_1}(T)$$

Therefore any two integral ranges of T overlap.

The integrable functions and their integrals

Definition 10

A function T will be called integrable if and only if the inferior limit of the diameters of its integral ranges is zero.

Theorem 11

If T is integrable, then the intersection of the integral ranges of T is a single element I(T) of Ɓ.

We can choose a set of integral ranges $I_{\Delta_1}(T), I_{\Delta_2}(T), I_{\Delta_3}(T), \ldots$ of diameters <1,< $\frac{1}{2}, < \frac{1}{4}, \ldots$. Since these are closed and overlap, their intersection is a point. But since every integral range of T is closed and overlaps every $I_{\Delta_K}(T)$, this point is contained in every integral range of T.

Definition 12

The I(T) of Theorem 11 is called the integral of T over ϑ.

Theorem 13

If T is integrable then for every ε >0 corresponds a decomposition Δ under which the aggregate $\mu(\sigma_i)$ T (σ_i) is unconditionally summable and has a diameter <ε.

By definition $\rho\left(I_\Delta(T)\right) < \frac{\varepsilon}{2K}$ (where K is constant of quasy-norm).

Therefore $\rho\left(\sum_i \mu(\sigma_i)T(\sigma_i)\right) \le 2K\rho\left(Co\left(\sum_i \mu(\sigma_i)T(\sigma_i)\right)\right) \le 2K\rho\left(I_\Delta(T)\right) < \varepsilon$.

In our case, cannot always say, as in the case of functions with values in Banach spaces that, the function corresponding I(T,σ), for T integrable function on ϑ and σ∈Σ, is completely additive, but we can define that:

Definition 14

Quasy-norm of a integrable function call the number $\|T\| = \sup_{\sigma\in\Sigma} \|I(T,\sigma)\|$,which may be finite or +∞.

Theorem 15

If Δ is any decomposition of ϑ, function T is integrable over every set σ_i of composition of Δ, and the aggregate of the J(T, σ_i) is unconditionally summable, then T is integrable over ϑ and J(T) = $\sum_i I(T,\sigma_i)$.

Decompose each σ_i by a decomposition Δ_i ,under which $\left\|I_{\Delta_i}(T,\sigma_i) - I(T,\sigma_i)\right\| < \frac{\varepsilon}{(2K)^i}$ .Then the corresponding decomposition of ϑ will be summable, and its integrated range will be within a sphere of radius ε of $\sum_i I(T,\sigma_i)$.

Theorem 16

If T and U are integrable functions, and m is a real number, then mT and T+U are integrable, T(mT)=mI(T), and I(T+U)=I(T)+I(U).

Proof

The conclusions about mT are evident, since if $\rho(I_\Delta(T))<\varepsilon$, then $I_\Delta(mT)$ = $\overline{Co\left(\sum_i m\mu(\sigma_i)T(\sigma_i)\right)}$ = $\overline{mCo\left(\sum_i \mu(\sigma_i)T(\sigma_i)\right)}$ = $mI_\Delta(T)$ and so $\rho(I_\Delta(mT)) = \rho(mI_\Delta(T)) = \|m I_\Delta(T) - m I_\Delta(T)\| = |m|\rho(I_\Delta(T)) < m\varepsilon$.

Let see the function T+U=V. Since T and U are Birkhoff integrable, then for every ε>0 there are decompositions Δ and Δ_1 such that $\rho\left(I_\Delta(T)\right) < \varepsilon$ dhe $\rho\left(I_{\Delta_1}(U)\right) < \varepsilon$. So (see[2]), $I_{\Delta\Delta_1}(V) \subset I_\Delta(T) + I_{\Delta_1}(U)$ which is of diameter less than 2Kε.

Equalities I(mT)=mI(T) dhe I(T+U)=I(T)+I(U) immediately derived from the properties of convexity and closing of sets.

Corollary

For every two integrable functions T and U we can write:||mT||=|m|·||T|| dhe ||T+U|| ≤K (||T|| +||U||).

Proof

First,||I(mT,σ)||=||mI(T,σ)||=|m|·||I(T,σ)||≤|m|·||T|| and thus ||mT||≤|m|·||T||.

On the other hand |m|·||I(T,σ)||=||I(mT,σ)||≤||mT||.

For m ≠0,we have ||I(T,σ)||≤ $\frac{1}{|m|}$ ||mT|| and thus ||T||≤ $\frac{1}{|m|}$ |mT|| which is equivalent to |m|·||T||≤||mT||.

If m =0 then mT = 0∈Ɓ and so ||mT||=0, therefore |m|·||T||=0=||mT||.Thus the equality ||mT||= |m|·||T|| is true.

Since, for every two integrable functions T and U we have I(V)=I(U)+I(V) (where V=T+U), then I(V,σ) =I(T,σ) +I(V,σ). So that, for every σ∈ϑ, || I(V,σ)||≤K(||I(T,σ)|| +||I(V,σ)||)≤K(||T|| +||U||) and thus ||V||≤K(||T||+||U||).

Let be α:β→α(β) any linear transformation of quasy-Banach Ɓ into the quasy-Banach space U.

Theorem 17

If T is any integrable function of ϑ to Ɓ then, (i) the function U(p) = α(T(p)) is integrable (ii) I(U)= α(I(T)).

Proof

If T is summable under a decomposition Δ of ϑ, then so is U, and I_Δ(U)=αI(I_Δ(T)).This is true by definition of U for single terms μ(σ_i)U(σ_i). Since α is additive for finite sums, if we pass in limit, take the result I(U) = α(I(T)). (Because during passing in limit the ratio $\frac{\|\alpha(\beta)-\alpha(\beta')\|}{\|\beta-\beta'\|}$ is bounded).

The point (i) is clear, because integrable functions class is stable under multiplication with scalar and the sum of functions, while α is known that is linear. This completes the proof.

It is of interest be observed that is true a famous theorem on the Lebesgue integral case.

Theorem 18

If T_n is a Birkhoff integrable functions sequence of ϑ to Ɓ, $\mu(\vartheta)$ is finite and sequence T_n is uniformly convergent to the integrable function T then $I(T)=\lim_{n\to\infty} I(T_n)$.

Proof

Since T_n is a uniformly convergent sequence to a integrable function T then, there is a number n large enough such that, for every point p and for every Δ to have $\|T(p)-T_n(p)\|< \frac{\varepsilon}{3K\mu(\vartheta)}$.

On the other hand, the functions T_n are integrable and so, there is a number n large enough such that $\|I_\Delta(T_n)-I(T_n)\|<\frac{\varepsilon}{3K}$.

From the quasy-norm's property we can write:

$\|I(T_n)-I(T)\| \le K\ (\|I_\Delta(T_n)-I(T_n)\| + \|I_\Delta(T_n)-I(T)\|) \le K[\|I_\Delta(T_n)-I(T_n)\| + K\ (\|I_\Delta(T_n)-I_\Delta(T)\| + \|I_\Delta(T)-I(T)\|)]$.

Since $\|T(p)-T_n(p)\|< \frac{\varepsilon}{3K\mu(\vartheta)}$ derives that $\sup\left\|\sum_i \mu(\sigma_i)\left(T(p_i)-T_n(p_i)\right)\right\|<\frac{\varepsilon}{3K}$ because the remain term of convergent series tends to zero of space.

So $\|I(T_n)-I(T)\|$ tends to zero and $I(T)=\lim_{n\to\infty} I(T_n)$.

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