

Optimized Domain Decomposition Method for Non Linear Reaction Advection Diffusion Equation

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Abstract

This work is devoted to an optimized domain decomposition method applied to a non linear reaction advection diffusion equation. The proposed method is based on the idea of the optimized of two order (OO2) method developed this last two decades. We first treat a modified fixed point technique to linearize the problem and then we generalize the OO2 method and modify it to obtain a new more optimized rate of convergence of the Schwarz algorithm. To compute the new rate of convergence we have used Fourier analysis. For the numerical computation we minimize this rate of convergence using a global optimization algorithm. Several test-cases of analytical problems illustrate this approach and show the efficiency of the proposed new method.

Keywords: Non linear reaction advection diffusion equation, Modified fixed point method, Domain decomposition method, Optimized interface conditions, Finite volume discretization

Introduction

The aim goal of this paper is to propose an optimized domain decomposition method (DDM) to solve a non linear reaction advection diffusion equation on a bounded domain such that:

$$\begin{cases} F(u) + cu + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} - v\Delta u = f & \text{on } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad (1)$$

This kind of equations has been treated using domain decomposition methods by many autors these last two decades [1,2,4,5]. In the proposed work in [4,5], the autors have developed a new method based on DDM for

with which they proposed an approximated convergence rate corresponding to an optimized O02 interface conditions. The difficulty of this method is that the rate of convergence is not given explicitly, and its approximation is not given precisely.

In the article [3] we have developed more faster algorithm of Newton-Krylov-Schwarz applied to non stationary problem. Several authors have used many other ideas. For example in [8,9] we propose Newton or quasi-Newton type methods to solve the non linear system such that Broyden-Fletcher-Goldfarb-Shanno (BFGS) method.

These kinds of Newton methods can be used in the context of DDM, but numerically the convergence takes a long time for many non-linearity test-cases.

In the present work we suggest a new approach based on the fixed point theorem to solve the nonlinear partial differential equation of reaction-advection-diffusion that we meet most time in the fluid dynamics and environmental problems.

We propose to use the Schauder fixed point theorem to linearize the two sub-problems. We have calculated explicitly the rate of the optimized two order DDM using this technique and Fourier analysis. It is easy to optimize the rate of convergence using global optimization functions that are implemented in different softwares (Scilab, Matlab,..).

In this article we first present the method based on the fixed point theorem for which we prove the convergence and we show some numerical results obtained using the Scilab software.

In the second part of this work, we present the decomposition domain optimized order two suggested in [1,2] and we develop the technique for calculating explicitly the rate of convergence.

Several numerical examples show the efficiency of the proposed method.

The Modified Fixed point method

In this section we treat the stationary nonlinear reaction advection diffusion problem:

$$\begin{cases} F(u) + cu + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} - v\Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (2)$$

For a given one initial function u_0 , we construct a sequence of functions u_n such that:

$$\begin{cases} cu_{n+1} + a \frac{\partial u_{n+1}}{\partial x} + b \frac{\partial u_{n+1}}{\partial y} - v \Delta u_{n+1} = f - F(u_n) & \text{on } \Omega \\ u_{n+1} = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

In the following we show the convergence of u_n to the solution of problem (2).

Let $V = H_0^1(\Omega)$. Consider the application

$$\varphi: V \rightarrow V$$

$$v \mapsto u$$

such that u is the solution of the variational formulation:

$$\int_{\Omega} cuw + \int_{\Omega} \vec{a} \nabla uw + \int_{\Omega} \nabla u \nabla w = \int_{\Omega} (f - F(v))w \quad \forall w \in V \quad (4)$$

where

$$\vec{a} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Lemma 1.1 *if the assumptions:*

$$- \exists v \succ 0; c - \frac{1}{2} \operatorname{div} \vec{a} \geq 0; \text{ and } v \prec \mu$$

- a, b, c , and μ are bounded functions.

are verified, we assume that the variational formulation (4) has an unique solution u in V

Proof : Consider the bilinear form a :

$$a(u, w) = \int_{\Omega} cuw + \int_{\Omega} \vec{a} \nabla uw + \int_{\Omega} \mu \nabla u \nabla w$$

by the Holder and the Poincaré Inequalities a is continuous:

$$\begin{aligned} a(u, w) &\leq (\sup_{\Omega} |c|) \|u\| \|w\| + \sup_{\Omega} \|\vec{a}\| \|\nabla u\| \|w\| + \sup_{\Omega} \mu \|\nabla u\| \|\nabla w\| \\ &\leq (\sup_{\Omega} |c| + \sup_{\Omega} \|\vec{a}\| C_{\text{poincarre}} + C_{\text{poincarre}}^2 \sup_{\Omega} \mu) \|\nabla u\| \|\nabla w\| \end{aligned}$$

a is coercive:

$$\begin{aligned} \int_{\Omega} \vec{a} \nabla uw &= - \int_{\Omega} u \vec{a} \nabla w - \int_{\Omega} \operatorname{div} \vec{a} uw + \int_{\partial\Omega} \vec{n} \cdot \vec{a} uw \\ &= - \int_{\Omega} u \vec{a} \nabla w - \int_{\Omega} \operatorname{div} \vec{a} uw \end{aligned}$$

Thus,

$$\begin{aligned}
 a(u, u) &= \int_{\Omega} cuu + \frac{1}{2} \int_{\Omega} \bar{a} \nabla uu + \frac{1}{2} \int_{\Omega} \bar{a} \nabla uu + \int_{\Omega} \mu \nabla u \nabla u \\
 &= \int_{\Omega} cuu - \frac{1}{2} \int_{\Omega} u \bar{a} \nabla u - \frac{1}{2} \int_{\Omega} \operatorname{div} \bar{a} uu + \frac{1}{2} \int_{\Omega} \bar{a} \nabla uu + \int_{\Omega} \mu \nabla u \nabla u \\
 &= \int_{\Omega} (c - \frac{1}{2} \operatorname{div} \bar{a}) u^2 + \int_{\Omega} \mu \|\nabla u\|^2 \\
 &\geq \nu \|u\|^2
 \end{aligned}$$

The Lax Milgram theorem involves that there exists one solution of (2) named $\varphi(v)$

we have by subtraction:

$$a(\varphi(u) - \varphi(v), w) = \int_{\Omega} (F(u) - F(v))w \quad \forall w \in V$$

Theorem 1.1 Suppose that:

- $\exists \nu > 0; c - \frac{1}{2} \operatorname{div} \bar{a} \geq 0$ and $\mu > \nu$
- F is K -lipshitzian.
- c, a, b and μ are bounded functions and $\frac{K}{\mu} \ll 1$

Then the sequence $u_{n+1} = \varphi(u_n)$ converges to the unique solution of the nonlinear problem (2)

Proof. The application φ bellow is well defined by lemma 1.1

We take $w = \varphi(u) - \varphi(v)$ used in the coercivity of $a(.,.)$ in the proof of the lemma 1.1 then

$$\begin{aligned}
 a(\varphi(u) - \varphi(v), \varphi(u) - \varphi(v)) &= \int_{\Omega} (F(u) - F(v))(\varphi(u) - \varphi(v)) \\
 &\geq \nu \|\varphi(u) - \varphi(v)\|^2
 \end{aligned}$$

F is K -lipshitzian, then we have:

$$\nu \|\varphi(u) - \varphi(v)\|^2 \leq K \|u - v\| \|\varphi(u) - \varphi(v)\|$$

$\frac{K}{\nu} \ll 1$ then φ is a contraction, and so, applying the Schauder fixed point

theorem to the application φ we show that the equation $\varphi(u) = u$ have one solution and the sequence $u_{n+1} = \varphi(u_n)$ converges to this solution which is the solution of problem (3).

In the proof of the theorem we saw that:

$$\nu|\varphi(u)-\varphi(v)|^2 \leq K|u-v||\varphi(u)-\varphi(v)|$$

The convergence of the the fixed point algorithm is faster when K is increasingly small, so for obtaining a small coefficient K we propose to modify the problem (3) as follows:

$$\begin{cases} (c - \xi)u_{n+1} + \bar{a}\nabla u_{n+1} - \mu\Delta u_{n+1} = f - F(u_n) - \xi u_n & \text{on } \Omega \\ u_{n+1} = 0 & \text{on } \partial\Omega \end{cases} \quad (5)$$

The parameter ξ is added to the system (3) to construct a new sequence with a smaller Lipschitzian K coefficient.

Theorem 1.2 *Suppose that:*

- $\exists \nu > 0; c - \frac{1}{2} \operatorname{div} \bar{a} \geq 0$ and $\mu > \nu$
- c, a, b and μ are bounded
- $F \in C^2(\mathbb{R})$
- $\xi(x) = F'(|x|)$

Then the sequence u_n of problem (3) converges to the solution of the nonlinear problem (1).

Proof. We assume that there exists a $\varepsilon < 1$ such that

$$|F(z) - F(y) - F'(|x|)(z - y)| < \varepsilon|z - y| \quad \forall y, z \in]x - \varepsilon, x + \varepsilon[\quad (F \in C^2(\mathbb{R}))$$

Consider $V = \{u \in H_0^1(\Omega) \mid \|u - |x|\|_\infty \leq \varepsilon\}$ and let

$$\varphi : V \rightarrow V$$

$$v \mapsto u$$

such that u is the solution of the variational formulation:

$$\int_{\Omega} (c + F'(|x|))uw + \int_{\Omega} \bar{a}\nabla uw + \int_{\Omega} \mu\nabla u\nabla w = \int_{\Omega} (f + F'(|x|)v - F(v))w \quad \forall w \in H_0^1(\Omega)$$

As we have done for the proof of Theorem 1.1, we have: φ is well defined because of theorem 1.3 and we show that:

$$\begin{aligned} a(\varphi(u) - \varphi(v), \varphi(u) - \varphi(v)) &= \int_{\Omega} (F(u) - F(v) - F'(|x|)(u - v))(\varphi(u) - \varphi(v)) \\ &\geq \nu \|\varphi(u) - \varphi(v)\|^2 \end{aligned}$$

Thus

$$\nu|\varphi(u)-\varphi(v)|^2 \leq \varepsilon|u-v||\varphi(u)-\varphi(v)|$$

and then φ is continuous, so the Schauder theorem involves that the sequence u_n of problem (3) converges to the unique solution of the weak formulation of the nonlinear problem (1) \square

In the next theorem we generalize the lemma (1.1) for all functions a, b and c for the linear problem:

$$\begin{cases} cu + \vec{a}\nabla u - \mu\Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (7)$$

Notice that the first condition of lemma (1.1) is omitted in the next theorem.

Theorem 0.3 *Suppose that:*

- $\exists v > 0; \mu(x) \geq v \quad \forall x \in \Omega$
- c, a, b and μ are given arbitrary bounded functions.

Then the problem (4) has an unique solution in $H_0^1(\Omega)$.

Proof. We can choose function C_0 and two parameters ε and μ_0 such that:

- $\frac{\mu_0}{\varepsilon\mu + \mu_0} < \frac{1}{2}$;
- $C_0 + \varepsilon(c - \frac{1}{2} \operatorname{div} \vec{a}) \geq 0$
- $\frac{\varepsilon\mu + \mu_0}{2} \geq C_0 \geq 0$

So consider:

$$\varphi : V \rightarrow V$$

$$v \mapsto u$$

such that u is the solution of the variational problem:

$$\int_{\Omega} (C_0 + \varepsilon c)uw + \int_{\Omega} \varepsilon \vec{a}\nabla uw + \int_{\Omega} (\mu + \varepsilon\mu_0)\nabla u\nabla w = \int_{\Omega} (f + C_0)vw + \dots + \int_{\Omega} \mu_0\nabla v\nabla w \quad \forall w \in V$$

(8)

As we have done for the proof of Theorem 1.1 we have: φ is well defined because

$$c + C_0 - \frac{1}{2} \operatorname{div} \vec{a} \geq 0$$

and we show that:

$$a(\varphi(u) - \varphi(v), \varphi(u) - \varphi(v)) = \int_{\Omega} (C_0(u - v)(\varphi(u) - \varphi(v)) + \dots$$

$$\dots \mu_0 \int_{\Omega} \nabla(u - v)\nabla(\varphi(u) - \varphi(v))) \geq (\varepsilon v + \mu_0) \|\varphi(u) - \varphi(v)\|^2$$

Thus

$$(\varepsilon\nu+\mu_0)|\varphi(u)-\varphi(v)|^2\leq(C_0+\mu_0)|u-v||\varphi(u)-\varphi(v)|$$

and then φ is continuous, so by the Schauder theorem φ has one fixed point which is the solution of problem (4) \square

Remark 1 To accelerate the fixed point of problem (1) we propose the sequence u_n such that:

$$\begin{cases} (c - \xi)u_{n+1} + a\nabla u_{n+1} - \mu\Delta u_{n+1} = f - F(u_n) - \xi u_n & \text{on } \Omega \\ u_{n+1} = 0 & \text{on } \partial\Omega \end{cases} \quad (9)$$

we take for example

$$\xi = F'(\varphi(x))$$

$$\xi = F'(u_n)$$

$$\xi = \frac{F(u_n) - F(0)}{u_n}$$

Remark 2 With the same way we can apply this concept of fixed point to the general following problem:

$$\begin{cases} \frac{\partial u}{\partial t} + F(u) + A(u)\nabla u - \mu(u)\Delta u & \text{on } \Omega \times [0, T] \\ u = g & \text{on } \partial\Omega \times [0, T] \\ u(t = 0) = h \end{cases} \quad (10)$$

where A and B and μ are non linear functions. Indeed:

For a given initial function U_0 , we construct iteratively a sequence function u_n as follows:

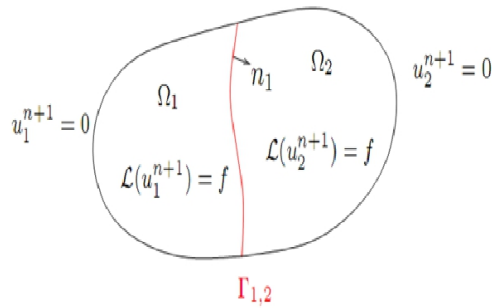
$$\begin{cases} \frac{\partial u_{n+1}}{\partial t} + cu_{n+1} + A(u_n)\nabla u_{n+1} - \mu(u_n)\Delta u_{n+1} = f - F(u_n) & \text{on } \Omega \times [0, T] \\ u_{n+1} = g & \text{on } \partial\Omega \times [0, T] \\ u_{n+1}(t = 0) = h \end{cases} \quad (11)$$

Domain decomposition with optimized interface of two order(DDM OO2):

The use of finite volumes , finite differences or finite elements solvers on high order meshes requires a high cost of computation.

Domain decomposition methods can reduce this cost by splitting initial problem into two or more sub-problems with smaller dimensions. Many authors have studied domain decomposition methods these last decades [10, 11]. Among these methods we consider in this work the method called second order optimized method OO2. This method was developed by different authors [1,2]. The main idea of the OO2 technique is described briefly as follows:

We split the domain Ω in two sub-domains Ω_1 and Ω_2 with an interface Γ (see figure 1)



then we built two sequences u_1^p and u_2^p respectively solutions of two sub-problems as described bellow:

we choose two initials functions u_1^0 defined on Ω_1 and u_2^0 defined on Ω_2

then we consider the two problems:

$$\begin{cases} L(u_1^{p+1}) = f & \text{on } \Omega_1 \\ u_1^{p+1} = g & \text{on } \partial\Omega \cap \Omega_1 \\ B_1(u_1^{p+1}) = B_1(u_2^p) & \text{on } \Gamma \end{cases} \quad (12)$$

And

$$\begin{cases} L(u_2^{p+1}) = f & \text{on } \Omega_2 \\ u_2^{p+1} = g & \text{on } \partial\Omega \cap \Omega_2 \\ B_2(u_2^{p+1}) = B_2(u_1^p) & \text{on } \Gamma \end{cases} \quad (13)$$

Where

$$L(u) = cu + \vec{a}\nabla u - \mu\Delta u \quad (14)$$

$$\begin{cases} B_1(u) = \frac{\partial u}{\partial n} - C_1 u + C_2 \frac{\partial u}{\partial \tau} - C_3 \frac{\partial^2 u}{\partial \tau^2} \\ B_2(u) = -\frac{\partial u}{\partial n} - (C_1 - \frac{a}{\mu})u + C_2 \frac{\partial u}{\partial \tau} - C_3 \frac{\partial^2 u}{\partial \tau^2} \end{cases} \quad (15)$$

n and τ are the normal and the tangent on Ω_1

Because of the Fourier analysis we show that the rate of convergence in the fourier way is (see [6] for proof)

$$\rho(C_1, C_2, C_3, k) = \left(\frac{\lambda^-(k) - C_1 + ikC_2 + C_3k^2}{\lambda^+(k) + \frac{a}{\mu} - C_1 + ikC_2 + C_3k^2} \right)^2 \quad (16)$$

where

$$\lambda^\pm(k) = \frac{a \pm \sqrt{a^2 + 4c\mu - 4ik\mu b + 4k^2\mu^2}}{2\mu}$$

In the following we resume the demonstration in the case of which the interface Γ is the line $x=0$, then the normal and the tangent vector are respectively $n=\vec{i}$ and $\tau=\vec{j}$

we have by substraction:

$$L(u_1^{p+2} - u) = 0 \quad \text{and} \quad L(u_2^{p+1} - u) = 0$$

$$B_1(u_1^{p+2} - u) = B_1(u_2^{p+1} - u); \quad \text{and} \quad B_2(u_2^{p+1} - u) = B_2(u_1^p - u)$$

u is the exact solution of the problem . if ω^{p+2} is the fourier transform of $u_1^{p+2} - u$, we have

$$c\omega^{p+2} + a \frac{\partial \omega^{p+2}}{\partial x} - ibk\omega^{p+2} - \mu \frac{\partial^2 \omega^{p+2}}{\partial x^2} + k^2 \omega^{p+2} = 0$$

k is the fourier frequency. If we take a solution in the form of $\omega(0, k)e^{\lambda x}$ we have

$$c+a\lambda-ibk-\mu\lambda^2+\mu k^2=0$$

it's an equation of two order which have two solution

$$\lambda^\pm(k) = \frac{a \pm \sqrt{a^2 + 4c\mu - 4ik\mu b + 4k^2\mu^2}}{2\mu}$$

since $\lim_{x \rightarrow -\infty} \omega^{p+2}(x, k) = 0$ we take $\omega^{p+2}(x, k) = \omega^{p+2}(0, k)e^{\lambda^+ x}$ (remark that

$$\lambda^+ > 0 \quad \text{and} \quad \lambda^- < 0)$$

by the same way if ω^{p+1} is the fourier transform of $u_2^{p+1} - u$, we have

$$\omega^{p+1}(x, k) = \omega^{p+1}(0, k)e^{\lambda^- x}$$

The conditions

$$B_1(\omega^{p+2}(0, k)) = B_1(\omega^{p+1}(0, k)) \quad \text{and} \quad B_2(\omega^{p+1}(0, k)) = B_2(\omega^p(0, k))$$

implies

$$(\lambda^+(k) - C_1 + ikC_2 + C_3k^2)\omega^{p+2}(0, k) = (\lambda^-(k) - C_1 + ikC_2 + C_3k^2)\omega^{p+1}(0, k)$$

$$(\lambda^-(k) - C_1 + \frac{a}{\mu} + ikC_2 + C_3k^2)\omega^{p+1}(0, k) = (\lambda^+(k) - C_1 + \frac{a}{\mu} + ikC_2 + C_3k^2)\omega^p(0, k)$$

Thus the rate of convergence is

$$\rho(C_1, C_2, C_3, k) = \frac{\omega^{p+2}(0, k)}{\omega^p(0, k)} = \frac{\lambda^-(k) - C_1 + ikC_2 + C_3k^2}{\lambda^+(k) - C_1 + ikC_2 + C_3k^2} \times \frac{\lambda^+(k) - C_1 + \frac{a}{\mu} + ikC_2 + C_3k^2}{\lambda^-(k) - C_1 + \frac{a}{\mu} + ikC_2 + C_3k^2}$$

Since $\lambda^+ + \lambda^- = \frac{a}{\mu}$, we have expression (11) The next theorem

show the convergence of OO2 method as developed in [6]

Theorem 2.4 Suppose that $c > 0$ and $\text{sign}(b) = \text{sign}(C_2)$ and $C_3 \geq 0$ then,

$$\max_{|k| < \frac{\pi}{h}} \|\rho(C_1, C_2, C_3, k)\| < 1$$

Proof. see [6]

Remark 3 In order to optimize the method we need to optimize the rate of convergence so we look for:

$$\min_{C_1, C_2, C_3} \max_{|k| < \frac{\pi}{h}} \|\rho(C_1, C_2, C_3, k)\| < 1$$

For optimizing this rate we have implemented the global optimization method [7]. Notice that this last problem have at least two optimums.

Remark 4 The OO2 method converges if $c > 0$. It can be generalized to a problem with coefficients a and b are variables. Nevertheless, if c is negative the method is not convergent in the most cases of this kind of equations.

In the next theorem we prove the convergence this problem when $c < 0$:

Theorem 0.5 Let $c < 0$ then the problem (12) and (13) converge when applying the OO2 method

Proof. If $c < 0$ then $1 - c > 0$, so applying the following fixed point

$$\begin{cases} (1 - c)u_{n+1} + \vec{a}\nabla u_{n+1} - \mu\Delta u_{n+1} = f - F(u_n) - (2c - 1)u_n \\ u_{n+1} = g \quad \text{on } \partial\Omega \end{cases} \quad (17)$$

the OO2 algorithm converges because $1-c>0$. □

We take the generalized artificial coefficients:

$$\left\{ \begin{array}{l} B_1'(u) = \frac{\partial u}{\partial n} - C_1 u + C_2 \frac{\partial u}{\partial \tau} - C_3 \frac{\partial^2 u}{\partial \tau^2} + \frac{\partial u}{\partial n} - C_1 u(0, y-a) + C_2 \frac{\partial u(0, y-a)}{\partial \tau} - C_3 \frac{\partial^2 u(y-a)}{\partial \tau^2} \\ B_2(u) = -\frac{\partial u}{\partial n} - (C_1 - \frac{a}{\mu})u + C_2 \frac{\partial u}{\partial \tau} - C_3 \frac{\partial^2 u}{\partial \tau^2} - (C_1 - \frac{a}{\mu})u(0, y+a) + C_2 \frac{\partial u(0, y+a)}{\partial \tau} - C_3 \frac{\partial^2 u(y+a)}{\partial \tau^2} \end{array} \right.$$

The Fourier transform of the the rate of convergence is calculated using:

$$F(f(y-a)) = \exp(ika)F(f)$$

The Coupled OO2 and Fixed point leads to the next subproblems:

$$\left\{ \begin{array}{l} L(u_1^{p+1}) = f - F(u_1^p) - \xi u_1^p \quad \text{on } \Omega_1 \\ u_1^{p+1} = g \quad \text{on } \partial\Omega \cap \Omega_1 \\ B_1(u_1^{p+1}) = B_1(u_1^p) \quad \text{on } \Gamma \end{array} \right. \quad (18)$$

and

$$\left\{ \begin{array}{l} L(u_2^{p+1}) = f - F(u_2^p) - \xi u_2^p \quad \text{on } \Omega_2 \\ u_2^{p+1} = g \quad \text{on } \partial\Omega \cap \Omega_2 \\ B_2(u_2^{p+1}) = B_2(u_1^p) \quad \text{on } \Gamma \end{array} \right. \quad (19)$$

where:

$$L(u) = (c - \xi)u + \bar{a}\nabla u - \mu\Delta u \quad (20)$$

Remark 5 For the method consisting in the resolution of the partial differential equation (1.1) by the finite volumes method, or finite differences method and solving the obtained non linear system using Newton, the convergence rate depends on the solution of partial differential equation. So it's impossible to compute this rate of convergence. One idea is to suggest a non linear decomposition [4,5] by eliminating the terms that we don't know in the convergence rate. Unfortunately, this method is often inaccurate.

III. Finite volume discretization:

To apply the finite volumes method, we choose a squared control volume C (figure (2))

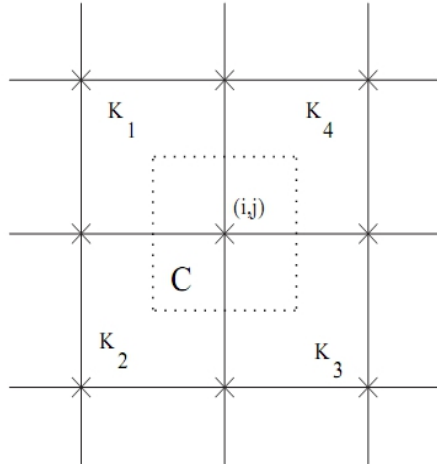


Figure 2: Control volume and we integer partial differential equation on C:

$$\int_C cu + \int_C a \frac{\partial u}{\partial x} + \int_C b \frac{\partial u}{\partial y} - \int_C \mu \Delta u = \int_C f$$

We approximate each member of this equation(using classical Finite volume method):

$$\int_C cu \approx \frac{u_{i,j} C_{i,j}}{4} \sum_{l=1}^4 Vol(K_l)$$

$$\int_C \vec{a} \nabla u \approx \frac{1}{4} \sum_{l=1}^4 \int_{K_l} \vec{a} \nabla u \approx \frac{1}{4} \sum_{l=1}^4 \int_{\partial K_l} \vec{a} \vec{n} u$$

$$\int_C \mu \Delta u = \int_{\partial C} \mu \frac{\partial u}{\partial n}$$

if the control volume C is on the interface Γ (figure (3)), we integer the partial differential equation on both $CU\Omega_1$ and $CU\Omega_2$.

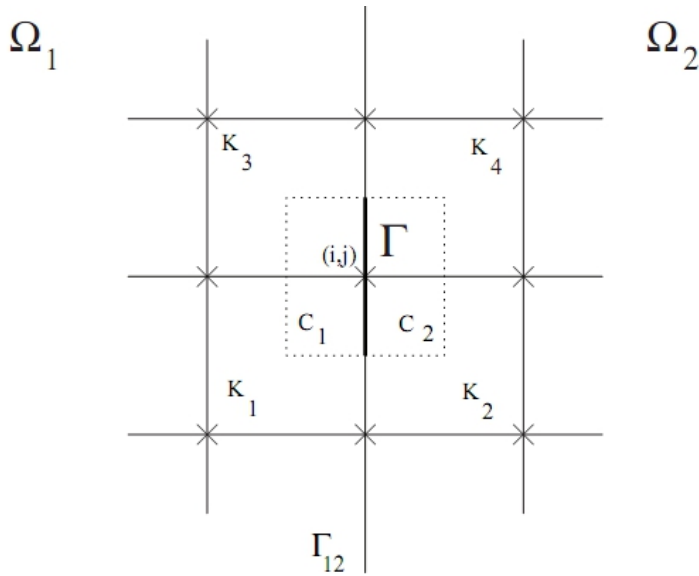


Figure 3: Control volume which include a part of the interface

So we have at iteration $p+1$ of the method (7) and (8):

$$\int_{C_1} L(u_1^{p+1}) + \int_{C_2} L(u_2^p) = \int_{C_1} f + \int_{C_2} f$$

where,

$$\int_{C_i} L(u) = \int_{C_i} cu + \int_{C_i} \bar{a} \nabla u - \int_{C_i} \mu \Delta u$$

the terms $\int_{C_i} cu + \int_{C_i} \bar{a} \nabla u$ are calculated as presented before. and we have:

$$\int_{C_i} \mu \Delta u = \int_{C_i/\Gamma} \mu \frac{\partial u}{\partial n_i} + \int_{\Gamma} \mu \frac{\partial u}{\partial n_i}$$

Replacing in (12):

$$\begin{aligned} \int_{C_1} cu_1^{p+1} + \int_{C_1} \bar{a} \nabla u_1^{p+1} + \int_{C_2} cu_2^p + \int_{C_2} \bar{a} \nabla u_2^p + \int_{C_1/\Gamma} \mu \frac{\partial u_1^{p+1}}{\partial n_1} + \int_{C_2/\Gamma} \mu \frac{\partial u_2^p}{\partial n_2} + \\ \dots + \int_{\Gamma} \mu \left(\frac{\partial u_1^{p+1}}{\partial n_1} - \frac{\partial u_2^p}{\partial n_2} \right) = \int_{C_1} f + \int_{C_2} f \end{aligned}$$

because $B_1(u_1^{p+1}) = B_1(u_2^p)$ we have:

$$\frac{\partial u_1^{p+1}}{\partial n_1} - \frac{\partial u_2^p}{\partial n_2} = (C_1 - C_2 \frac{\partial}{\partial \tau_1} + C_3 \frac{\partial^2}{\partial \tau_1^2})(u_1^{p+1} - u_2^p)$$

so we replace this equality in the last equation to complete the interface condition and to ensure the convergence of the finite volumes method.

Numerical simulation

Numerically, we have implemented the finite volumes method to approach the sub-problems obtained after applying the fixed point algorithm.

We obtain a good accuracy, the error is in the order of 10^{-5} forward the third iteration on some usual test-functions .We take for F respectively the expressions

$$F(u)=u^p$$

$$F(u)=e^u$$

$$F(u)=\frac{a}{b+u}$$

$$F(u)=\log(1-u)$$

The following results show the error between the approximate solution using the modified fixed point method and the exact solution $u_{exact}=xy(1-x)(1-y)$ of the problem (1). $Error = \|u - u_{exact}\|_{L^2}$

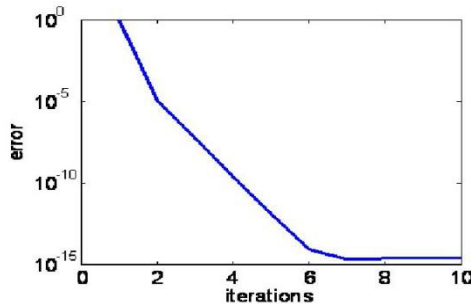


Figure 4: The error: $c=\mu=1, a=b=0, h=0.025; F(u)=u*u$

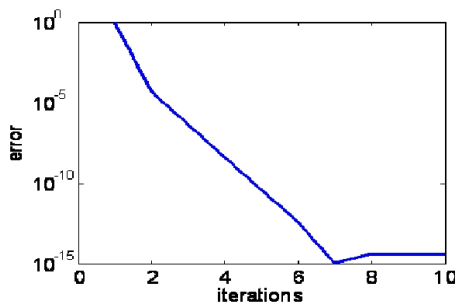


Figure 5: The error: $c=2, \mu=0.1, a=-1, b=0, h=0.001$ and $F(u)=\exp(u)$

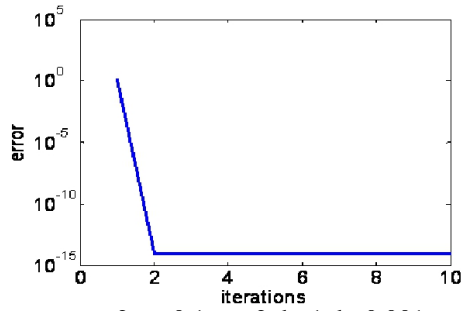


Figure 6: The error: $c=2, \mu=0.1, a=-2, b=1, h=0.001$ and $F(u)=a/(b+u)$

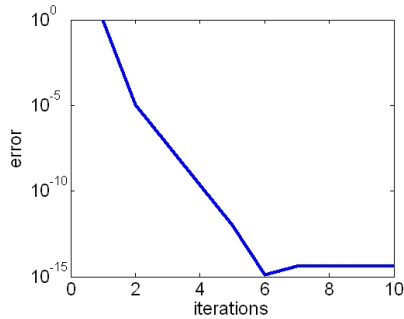


Figure 7: $c=-1, \mu=0.5, a=2, b=-1.5, h=0.001$ and $F(u)=\log(1-u)$

h is the mesh grid.

Now, we show the results of combining modified fixed point and OO2 algorithm.

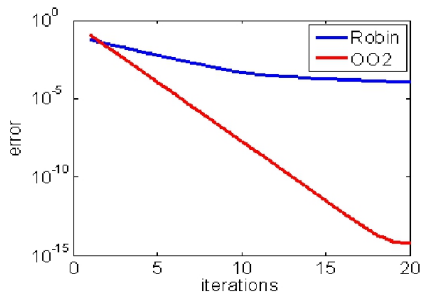


Figure 8: case1

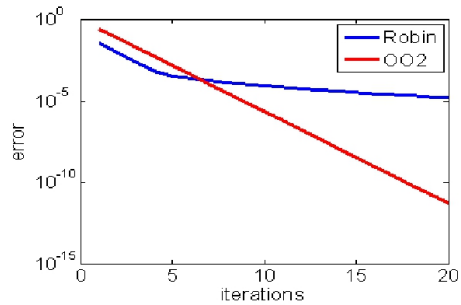


Figure 9: case2

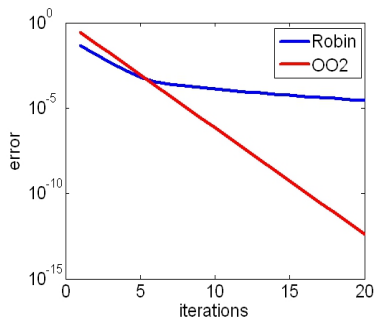


Figure 10: case3

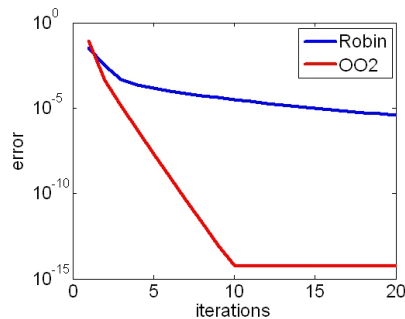


Figure 11: case4

The figures 8,9,10,11 showing the L^2 error between the combining modified fixed point solution and the OO2 algorithm and the exact solution.

Test case1: $c=0.1$, $\mu=1$, $a=b=-2$, $h=0.025$, $F(u)=\log(u)$ and $u_{exact}=xy(1-x)(1-y)$.

Test case2: $c=2$, $\mu=0.1$, $a=-1$, $b=0$, $h=0.001$, $F(u)=u^2$ and $u_{exact}=\exp(xy(1-x)(1-y))-1$.

Test case3: $c=-2$, $\mu=0.1$, $a=-2$, $b=1$, $h=0,001$, $F(u)=u^3$ and $u_{exact}=\exp(xy(1-x)(1-y))-1$.

Test case4: $c=-1$, $\mu=0.5$, $a=\sin(xy)$, $b=-\cos(xy)$, $h=0,001$, $F(u)=\exp(u)-1$ and $u_{exact}=xy(1-x)(1-y)$.

h is the mesh grid

Here are some examples of results of the proposed algorithm corresponding to some tests functions

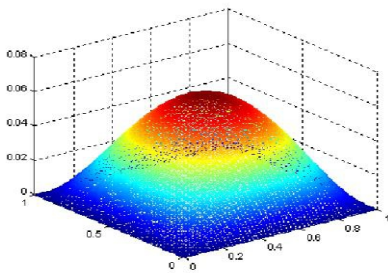


Figure 12: case1.

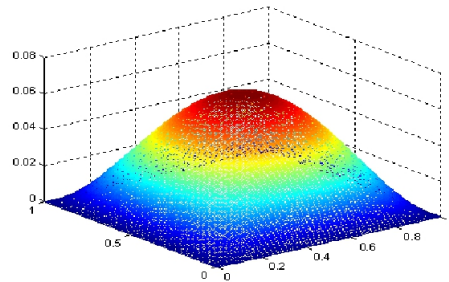


Figure 13: case2.

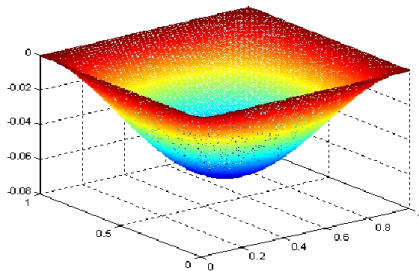


Figure 14: case3

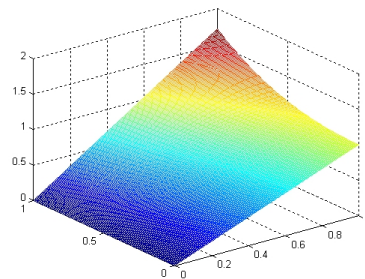


Figure 15: case4

case1: $u=xy(1-x)(1-y)$

case2: $u=\exp(xy(1-x)(1-y))-1$.

case3: $u=\ln(1-xy(1-x)(1-y))$.

case4: $u=\exp(x*y)-xy$

Conclusion

In this work we have developed an optimized domain decomposition algorithm applied to a non linear PDE. We firstly have proposed a proof of

the convergence of the fixed point technique applied to the non linear equation. We have proposed a new approach for computing the convergence rate using the Fourier analysis and global optimization. Secondly we have presented several test-cases to show the efficiency of this approach. The fundamental result is that we obtained a well optimized rate of convergence of the proposed algorithm in comparison with global calculation using classical solvers. As perspective of the present work, we can study the following ideas:

- Generalize the method to non linear and non stationary equation
- Generalize the approach to the nonlinear diffusion equation.
- Apply the method to real problems in fluid dynamics, environmental sciences or the image processing.

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