GLOBAL STABILITY AND SPATIAL SPREAD OF THE NONLINEAR EPIDEMIC MODEL

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Abstract

In this paper, we consider a non linear epidemic model SIQS by experiencing the disease; whenever infected, the disease individuals will return to the susceptible class after a fixed period of time. First, the local stabilities and global stability of the infection-free equilibrium and endemic equilibrium are analyzed, respectively. Second, the endemic equilibrium is formulated in terms of the incidence rate, and after we stydy local and global asymptotic stability. We study the stochastique systme by pertubing the contact rate, and we mainly use the theorie of Itô formula. Finnaly we study the spatial spread model with traveling wave solution.

Keywords: Basic reproductive number, epidemic model, global stability,
local asymptotically stable

Introduction

Classical epidemic models assume that the size of the total population is constant. More recent models consider a population size variable to take into account a longer period with death and disease causing reduced reproduction. Generally, a model contains a disease-free equilibrium and one or multiple equilibria are endemic. The stability of a disease-free status equilibrium and the existence of other nontrivial equilibria can be determine by the ratio called the basic reproductive number, which quantifies the number of secondary infections arise from infected in a population of sensitive.

In recent years the dynamics of the non linear epidemic models have received considerable attention in the refferences, in order to describe the effects of disease in differents models. Motivated by the comment of the differents autors In this paper, we

discuss the equilibrium and global , local stability of the non linear SIQS epidemic model with constant paramters. We have made the following contributions:

1. The local and global stabilities of the infection-free equilibrium are analysed, respectively in section 3.
2. Second the endemic equilibrium is formuled in terms of the insidence rate and the differents positive parametres and lacally, globallay asymptotic stabilities are found in section 4 and 5.
3. We study the sochastic system by pertubieng the contact rate in section 6.
4. Finally we considere the spatial spread of the ifectives and susceptibles in the section 7 then we analyse the system we travelling the wave solution in section 8.

Model equations

This paper considers the following SIQS nonlinear epidemic model:

$$\begin{cases} \dot{S}(t) = (\mu + \gamma - \mu_1 - \beta)S(t) + \alpha I(t) - kS(t)I(t) + \nu, \\ \dot{I}(t) = \beta S(t) - (\mu_2 + \gamma + \alpha)\, I(t) + kS(t)I(t) + \rho, \\ \dot{Q}(t) = \gamma I(t) - \gamma S(t) - \mu_3 Q(t) \end{cases} \tag{2.1}$$

Where S (t) + I (t) + Q (t) =N (t) denotes the sizes of the population at time t; S (t), I (t) and Q (t) denote the sizes of the population susceptible to disease, of infective members, and members who have been in quarantine with the possibility of infection, respectively. It is assumed that all newborns are susceptible. The positive constants μ_1, μ_2, and μ_3 represent the death rates of susceptible, infectivity and quarantine, respectively. Biologically, it is natural to assume that $\mu_1 \le \min\{\mu_2, \mu_3\}$. The positive constants μ and γ represent the birth rate (incidence rate) of the population and the recovery rate of infective, respectively. The positive constants α , β are the average numbers of contacts infective for S and I. The term γ S, indicate that an individual has survived from natural death in a recovery before becoming susceptible again. The constant k is the rate of unknown members infected which is detected by the system. The positive constants ν, ρ are the parameters of immigrations.

The initial condition of (2.1) is given as

$$S(\eta) = \varnothing_1(\eta), I(\eta) = \varnothing_2(\eta), Q(\eta) = \varnothing_3(\eta), -\tau \le t \le 0, \tag{2.2}$$

Where $\varnothing = (\varnothing_1, \varnothing_2, \varnothing_3)^T \in \square$, such that:

$$S(\eta) = \varnothing_1(0) = S_0, I(\eta) = \varnothing_2(0) = I_0, Q(\eta) = \varnothing_3(0) = Q_0. \tag{2.3}$$

Let *C* denotes the Banach space $C([-\tau, 0], R^3)$ of continuous functions mapping the interval $[-\tau, 0]$ into R^3, with τ is latent time.

Consider the system without the parameters of immigration and study the stability of the system, and since *Q* (*t*) does not appear explicitly in the first two equations of system (2.1), which is positive it means that the solution remains positive for any trajectory initialized to positive conditions. We consider the system:

$$\begin{cases} \dot{S}(t) = (\mu + \gamma - \mu_1 - \beta)S(t) + \alpha I(t) - kS(t)I(t), \\ \dot{I}(t) = \beta S(t) - (\mu_2 + \gamma + \alpha)\, I(t) + kS(t)I(t), \end{cases} \tag{2.4}$$

With the initial conditions (2.2), (2.3)

Since $\dot{N} \le -\mu_1 N$, and by integration we get:

$N \le N_0 e^{-\mu_1 t}$, for all t≥0 (2.5)

With initial conditions:

$S(0) = S_0, I(0) = I_0, Q(0) = Q_0$, and $N_0 = S_0 + I_0 + Q_0$. (2.6)

Then we consider the system only in the region $\Omega=\{(S,I) \in R_+^2, S+I \le N \prec \frac{\mu S}{\mu_1}\}$, is positively invariant set of (2.4).

The disease-free equilibrium and its stability.
Local stability

An equilibrium point of system (2.4) satisfies

$$\begin{cases} (\mu + \gamma - \mu_1 - \beta)S + \alpha I - kSI = 0, \\ \beta S - (\mu_2 + \gamma + \alpha)\, I + kSI = 0, \end{cases} \tag{3.1}$$

It can be seen that, whenever all the seven associated parameters assume positive values, system (3.1) has a disease-free equilibrium of the form $E_0 = (0,0)^T$.

We start by analyzing the behavior of the system (2.4) near E_0.The characteristic equation of the linearization of (2.4) near E_0 is:

$$\det \begin{pmatrix} (\mu + \gamma - \mu_1 - \beta) - A & \alpha \\ \beta & -(\mu_2 + \gamma + \alpha) - A \end{pmatrix} = 0 \tag{3.2}$$

E_0, is locally asymptotically stable if and only if the trace of the Jacobin matrix near E_0 is strictly negative and its determinant is strictly positive.

E_0, is locally asymptotically stable $\Leftrightarrow \begin{cases} \mu-(\mu_1+\mu_2)-(\alpha+\beta)<0 \\ (\mu_1+\beta-\mu-\gamma)(\mu_2+\gamma+\alpha)-\alpha\beta>0 \end{cases}$ (3.3)

As long as condition (3.3) holds the disease-free equilibrium of system (2.4) is unique and stays locally asymptotically stable.

Let us define the basic reproduction number of the infection as

$$R_0 = \frac{\alpha\beta}{(\mu_1+\beta)(\mu_2+\alpha+\beta)} + \frac{\mu+\gamma}{\mu_1+\beta} \quad (3.4)$$

Lemma

If R_0 <1, then the disease-free point E_0 is locally asymptotically stable; Stable if R_0 =1, and unstable if R_0 >1.

Global stability

Theorem 1

If R_0 <1, then the disease-free point E_0 is globally asymptotically stable in Ω.

Proof

Let $(S_0, I_0) \in \Omega$ .Since $\dot{I}(t) = \beta S(t) - (\mu_2 + \gamma + \alpha)\, I(t) + kS(t)I(t)$,

Then $\dot{I}(t) \le -(\mu_2 + \gamma + \alpha)\, I(t)$ so by integration, we get

$I(t) \le I_0 e^{-(\mu_2+\gamma+\alpha)t}$, for every t≥0 (3.5)

If R_0 <1 then $(\mu_2+\gamma+\alpha)$ >0. Hence, I (t) converges to zero.

The second equation from (2.4) gives

$$\dot{S} = (\mu+\gamma-\mu_1-\beta)S + \alpha I - kSI \le (\mu+\gamma-\mu_1-\beta)S + \alpha I_0 e^{-(\mu_2+\gamma+\alpha)t} \quad (3.6)$$

Integrating the above inequality, we obtain

$$S(t) \le S_0 e^{-(\mu_1+\beta-\mu-\gamma)t} + \frac{\alpha I_0}{\left[(\mu_1+\beta-\mu-\gamma)-(\mu_2+\gamma+\alpha)\right]}\left(e^{-(\mu_2+\gamma+\alpha)t} - e^{-(\mu_1+\beta-\mu-\gamma)t}\right), \quad (3.7)$$

$$S(t) \le \left(S_0 + \frac{\alpha I_0}{\left[(\mu_1+\beta-\mu-\gamma)-(\mu_2+\gamma+\alpha)\right]}\right)e^{-mt}, \quad (3.8)$$

With

$$m = \min\{(\mu_1+\beta-\mu-\gamma), (\mu_2+\gamma+\alpha)\} \quad (3.9)$$

So if R_0 <1, S (t) converges to zero.□

Lemma2. [7]

Let D be a bounded interval in $\mathbb{R}$ *and h:* $(t_0, \infty) \times D \rightarrow \mathbb{R}$, be bounded and uniformly continuous function, *let x:* $(t_0, \infty) \times D \rightarrow \mathbb{R}$ be a solution:

$$\dot{x} = h(t, x) \tag{3.10}$$

Which is defined on the whole interval (t_0, ∞), then

$$\begin{aligned} &(i) \liminf_{t \to \infty} h(t, x_\infty) \le 0 \le \limsup_{t \to \infty} h(t, x_\infty), \\ &(i) \liminf_{t \to \infty} h(t, x^\infty) \le 0 \le \limsup_{t \to \infty} h(t, x^\infty), \end{aligned} \tag{3.11}$$

Where,

$$x_\infty = \liminf_{t \to \infty} x(t), x^\infty = \limsup_{t \to \infty} x(t). \tag{3.12}$$

Theoreme2.

Let $R_0=1$, then the disease-free point E_0 is globally asymptotically stable in Ω.

Proof

Let $(S_0, I_0, Q_0) \in \Omega$.

From the second equation to the system (2.2) we obtain $I(t) \le I_0 e^{-(\mu_2+\gamma+\alpha)t}$

Since $R_0=1$, *then* $\alpha\beta + (\mu+\gamma)(\mu_2+\alpha+\beta) = (\mu_1+\beta)(\mu_2+\alpha+\beta)$, and $\frac{dI}{dt} \le 0$.

So I (t) is a positive and non-increasing function, hence:

$$I(t) \overrightarrow{t \to \infty} l \in [0, \infty).$$

By the application of lemma2 to the third equation to the system (2.2), we get:

$$|\gamma I - \mu_3 Q_\infty|_\infty \le 0 \le |\gamma I - \mu_3 Q^\infty|^\infty \tag{3.13}$$

Therefore

$$\frac{\gamma l}{\mu_3} \le Q_\infty \le Q^\infty \le \frac{\gamma l}{\mu_3} \tag{3.14}$$

So

$$\lim_{t \to \infty} Q(t) \to Q_\infty = Q^\infty = \frac{\gamma l}{\mu_3} = Q(\infty) . \tag{3.15}$$

Applying the same method using lemma 2 we deduce that

$$\lim_{t\to\infty} S(t) = S(\infty) \tag{3.16}$$

Since $R_0=1$, *the point* E_0 *is* stable.

Hence for every ε>0, there exist η (ε) such that if

$$S_o + I_o + Q_o \le \eta(\varepsilon) \tag{3.17}$$

Then for every t≥0, S(t) ≤ε, I(t)≤ ε, et Q(t)≤ ε.

We get l≤ ε, for every ε>0 therefore l=0

Finally for all $(S_0, I_0) \in \Omega$

$$\lim_{t\to\infty} S(t) = S_o, \lim_{t\to\infty} I(t) = I_o, et \lim_{t\to\infty} Q(t) = Q_o. \square$$

Endemic equilibrium and its locally asymptotical stability

From the previous section it is follows that when the trivial equilibrium E_0 of system (3.1) is locally asymptotically stable, then the endemic equilibrium does not exist. When R_0 > 1, system (3.1) has a unique non-trivial equilibrium $E^* = (S^*, I^*)^T$ other than the disease-free equilibrium, where

$$E^* = \left(\frac{\mu_2+\alpha+\gamma}{k} - \frac{\beta(\mu_2+\gamma)}{k(\mu+\gamma-\mu_1)}, \frac{(\mu+\gamma-\mu_1)}{k} - \frac{\alpha(\mu+\gamma-\mu_1)}{k(\mu_2+\gamma)} - \frac{\beta}{k} \right)^T \tag{4.1}$$

By introducing $x(t) = S(t) - S^*, y(t) = I(t) - I^*,$ who are the small perturbations.

System (3.1) is centered at E^* and its linear part reads

$$\begin{cases} \dot{x}(t) = \left(\frac{\alpha(\mu+\gamma-\mu_1)}{\mu_2+\gamma} \right) x(t) + \left(\frac{(\beta+\mu_1-\mu-\gamma)(\mu_2+\gamma)}{\mu+\gamma-\mu_1} \right) y(t) \\ \dot{y}(t) = \left(\frac{(\mu+\gamma-\mu_1)(\mu_2+\gamma-\alpha)}{\mu_2+\gamma} \right) x(t) + \left(\frac{-\beta(\mu_2+\gamma)}{\mu+\gamma-\mu_1} \right) y(t) \end{cases} \tag{4.2}$$

The characteristic equation of (3.1) at E^* is

$$\det \begin{pmatrix} \left(\frac{\alpha(\mu+\gamma-\mu_1)}{\mu_2+\gamma} \right) - A & \frac{(\beta+\mu_1-\mu-\gamma)(\mu_2+\gamma)}{\mu+\gamma-\mu_1} \\ \frac{(\mu+\gamma-\mu_1)(\mu_2+\gamma-\alpha)}{\mu_2+\gamma} & \frac{-\beta(\mu_2+\gamma)}{\mu+\gamma-\mu_1} - A \end{pmatrix} = 0 \tag{4.3}$$

E^*, is locally asymptotically stable if and only if the trace of the matrix is strictly negative and its determinant is strictly positive.

$$E^*, \text{ is locally asymptotically stable} \Leftrightarrow \begin{cases} \frac{\alpha}{\beta} < 0 \\ (\mu_1 + \beta - \mu - \gamma)(\mu_2 + \gamma - \alpha) - \alpha\beta > 0 \end{cases} \quad (4.4)$$

The global stability for endemic equilibrium

Here, we restraint to the case when $\mu = \mu_1 = \mu_2$, we have in this case we have the system:

$$\begin{cases} \dot{S} = (\mu + \gamma - \mu_1 - \beta)S + \alpha I - kSI = P_1(S, I), \\ \dot{I} = \beta S - (\mu_2 + \gamma + \alpha) I + kSI = P_2(S, I), \end{cases} \quad (5.1)$$

Theorem3. Dulac Critere

We have D_1={S(t)>0, I(t)>0, S+I≤1}is the region connexe of plan to phase. If the function exist:

$$\frac{\partial(DP_1)}{\partial S} + \frac{\partial(DP_2)}{\partial I}, \dot{S} = P_1(S, I), \ \dot{I} = P_2(S, I) \quad (5.2)$$

The orbites in D_1 are not closes.

Theorem4.

If R_0> 1, then the endemique-disease point E^* is globally asymptotically stable.

Proof

Take a Dulac function [12], $D(S, I) = \frac{1}{SI}$ for S, I>0. We have

$$\frac{\partial(DP_1)}{\partial S} + \frac{\partial(DP_2)}{\partial I} = -\frac{\alpha}{S^2} - \frac{\beta}{I^2} < 0 \quad (5.3)$$

Hence, according to Dulac criterion, the system (2.4) has not periodic orbits. Since (5.3) admit only two equilibriums E_0 and E^*. When R_0> 1 and E_0 is unstable, hence by Poincar- Binedixon theorem [12], E^* is globally asymptotically stable. □

The stochastic system by perturbing the contact rate

We limit ourselves here to perturbing only the contact rate so we replace k by k+ σ W (t), where W(t) is white noise (Brownian motion). The system (2.4) is transformed to the following Itô stochastic differential equations:

$$\begin{cases} dS = \left[(\mu + \gamma - \mu_1 - \beta)S + \alpha I - kSI\right] - \sigma SIdW, \\ dI = \beta S - (\mu_2 + \gamma + \alpha) I(t) + kSI + \sigma SIdW, \\ dQ = \gamma I - \gamma S - \mu_3 Q \end{cases} \quad (6.1)$$

In this section, we will prove, under some conditions, that E_0 is globally exponentially mean square and almost surely stable, and for this purpose, we need the following Theorem:

Theorem5.

The set Ω is almost surely invariant by the stochastic system (6.1). Thus if $(S_0, I_0, Q_0) \in \Omega$, then $P\left[(S, I, Q) \in \Omega\right] = 1$.

Proof

The system (6.1) implies that $dN \leq -\mu_1 N dt$, hence $N \leq N_0 e^{-\mu_1 t}$, for all t≥0. Since $(S_0, I_0, Q_0) \in \Omega$

We have $I(t) \leq I_0 e^{-(\mu_2+\gamma+\alpha)t}$, for every t≥0, then $(\mu_2+\gamma+\alpha)>0$. Hence, $I(t)$ converges to zero.

Because, $S_0>0, I_0>0, and Q_0>0$, then there exist $\varepsilon_0>0$ such that $S_0>\varepsilon_0, I_0>\varepsilon_0, and Q_0>\varepsilon_0$.

Considering:

$$\tau_\varepsilon = \inf\left\{t \geq 0, S(t) \leq \varepsilon, I(t) \leq \varepsilon, Q(t) \leq \varepsilon\right\} for \varepsilon > \varepsilon_0, \tag{6.2}$$

$$\tau = \lim_{t\to 0} \tau_\varepsilon \inf\left\{t \geq 0, S(t) \leq 0, I(t) \leq 0, Q(t) \leq 0\right\}. \tag{6.3}$$

Let

$$V(t) = \log\frac{1}{S(t)} + \log\frac{1}{I(t)} + \log\frac{1}{Q(t)}. \tag{6.4}$$

Then, using Itô formula we have, for all t≥0 and $T \in [0, t \wedge \tau_\varepsilon]$,

$$\begin{aligned} dV(t) = &\left[\left(\mu_1+\beta-\mu-\gamma\right) - \alpha\frac{I(T)}{S(T)} + kI(T) + \frac{1}{2}I^2(T)\right]dT + \sigma I(T)dW(T) \\ &+\left[\left(\mu_2+\gamma+\alpha\right) - \beta\frac{S(T)}{I(T)} - kS(T) + \frac{1}{2}S^2(T)\right]dT - \sigma S(T)dW(T) \\ &+\left[\mu_3 + \gamma\frac{S(T)}{Q(T)} - \gamma\frac{I(T)}{Q(T)}\right]dT . \end{aligned} \tag{6.5}$$

$$\begin{aligned} dV(t) \leq &\left[\left(\mu_1+\mu_2+\mu_3+\alpha+\beta-\mu\right) + kI(T) + \frac{1}{2}I^2(T) + \frac{1}{2}S^2(T)\right]dT \\ &+\sigma\left(I(T) - S(T)\right)dW(T). \end{aligned} \tag{6.6}$$

For all $T \in [0, t \wedge \tau_\varepsilon]$, with $L = \mu_1 + \mu_2 + \mu_3 + \alpha + \beta - \mu$, therefore

$$dV(t) \le LdT + \sigma\left(I(T) - S(T)\right)dW(T). \tag{6.7}$$

Hence

$$V(t) \le LT + \sigma\int_0^T \left(I(u) - S(u)\right)dW(u). \tag{6.8}$$

With proposition 7.6 in [7] $\sigma\int_0^T \left(I(u) - S(u)\right)dW(u)$ is a mean zero process then:

$$E\left(V(t)\right) \le LT \tag{6.9}$$

For all t≥0, $EV(t \wedge \tau_\varepsilon) \le L(t \wedge \tau_\varepsilon) \le Lt$. (6.10)

Moreover $V(t \wedge \tau_\varepsilon) \ge 0, S(t \wedge \tau_\varepsilon) \ge 0, I(t \wedge \tau_\varepsilon) \ge 0, and Q(t \wedge \tau_\varepsilon) \ge 0$ thus,

$$EV(t \wedge \tau_\varepsilon) \ge EV(t) \times \chi_{[\tau_\varepsilon \le t]} + E\left(V(t) \times \chi_{[\tau_\varepsilon > t]}\right) \tag{6.11}$$

$$EV(t \wedge \tau_\varepsilon) \ge EV(t) \times \chi_{[\tau_\varepsilon \le t]} \ge P\left(\tau_\varepsilon \le t\right) \log \frac{1}{\varepsilon} \tag{6.12}$$

Where χ_D is the indicator function of a subset D. Combining (6.9) with (6.12) gives for all t≥0, $P(\tau_\varepsilon \le t) \le \frac{Lt}{\log \frac{1}{\varepsilon}}$. Tending ε to zero, we obtain for all t≥0, $P\left(\tau \le t\right) = 0$, from where $P\left(\tau \le \infty\right) = 0$.□

Spatial spread model

We consider the spatial spread of the infectives and susceptibles. By modeling the model (2.4), we adding the term S^2 represent the interaction between the same kind of species in the susceptible population. We obtain the following model:

$$\begin{cases} \dfrac{\partial S}{\partial t} = (\mu + \gamma - \mu_1 - \beta)S - kSI + \alpha I - \mu S^2 + d_1 \Delta S, \\ \dfrac{\partial I}{\partial t} = \beta S - (\mu_2 + \gamma + \alpha)\, I + kSI + d_2 \Delta I, \end{cases} \tag{7.1}$$

Where S=S(x,t) and I=I(x,t). ΔS , ΔI represent the diffusion of the infective and susceptible densities respectively.

Traveling wave solution

We seek a constant shape travelling wave solution of (7.1) by setting

$$S(x,t) = \tilde{S}(z), I(x,t) = \tilde{I}(z), \quad z = x - ct\ , \tag{7.2}$$

Where c is the wave speed, substitute (7.2) into (7.1) we get

$$\begin{cases} c\dfrac{\partial \bar{S}(z)}{\partial z} = (\mu_1 + \beta - \mu - \gamma)\bar{S}(z) + k\bar{S}(z)\bar{I}(z) - \alpha\bar{I}(z) + \mu\bar{S}^2(z) - d_1\dfrac{\partial^2 \bar{S}(z)}{\partial^2 z}, \\ c\dfrac{\partial \bar{I}(z)}{\partial z} = -\beta\bar{S}(z) + (\mu_2 + \gamma + \alpha)\bar{I}(z) - k\bar{S}(z)\bar{I}(z) - d_2\dfrac{\partial^2 \bar{I}(z)}{\partial^2 z}, \end{cases} \tag{7.3}$$

This assumption is a legitimate one because the infective population is very active in infecting other individuals in the total population and it is capable of moving more but the susceptibles is not so. Therefore we assume that d_1 is negligible compared to d_2, and then we analyze (7.3).

Hence, with $d_1 = 0$, and $\frac{\partial \bar{S}(z)}{\partial z} = \bar{S}'$ we can rewrite (7.3) as three ordinary differential equations

$$\begin{cases} \bar{S}' = \dfrac{(\mu_1 + \beta - \mu - \gamma)\bar{S} + k\bar{S}\bar{I} - \alpha\bar{I} + \mu\bar{S}^2}{c}, \\ \bar{I}' = \bar{T}, \\ \bar{T}' = \dfrac{(\mu_2 + \gamma + \alpha)\bar{I} - k\bar{S}\bar{I} - \beta\bar{S} - c\bar{T}}{d_2}, \end{cases} \tag{7.4}$$

In the $(\bar{S}, \bar{I}, \bar{T})$ phase space there are three steady states, namely

$$(0,0,0),\ (1 - \frac{\mu_1 + \beta - \gamma}{\mu}, 0, 0), (\hat{\bar{S}}, \hat{\bar{I}}, \hat{\bar{T}}) \tag{7.5}$$

With $(\hat{\bar{S}}, \hat{\bar{I}}, \hat{\bar{T}}) = \left((m_1 + m_2)e^{\left(1 - \frac{\mu_1 - \gamma}{\mu}\right)}, \frac{(\mu_2 + \gamma + \alpha)}{k} * \frac{1}{\hat{\bar{S}}} - \frac{\beta}{k}, 0 \right)$, m_1, m_2 are the constants . We expect the traveling wave front solution to be the form $(0,0,0) \text{to} (\hat{\bar{S}}, \hat{\bar{I}}, \hat{\bar{T}})$ and from

$(1 - \frac{\mu_1 + \beta - \gamma}{\mu}, 0, 0) \text{to} (\hat{\bar{S}}, \hat{\bar{I}}, \hat{\bar{T}})$

Therefore we have to seek solution $\left(\bar{S}, \bar{I}\right)$ of (7.4) with the following boundary conditions:

$$\bar{S}(-\infty) = 0, \bar{I}(-\infty) = 0, \bar{S}(\infty) = \hat{\bar{S}}, \bar{I}(\infty) = \hat{\bar{I}}, \tag{7.6}$$

$$\bar{S}(-\infty) = 1 - \frac{\mu_1 + \beta - \gamma}{\mu}, \bar{I}(-\infty) = 0, \bar{S}(\infty) = \hat{\bar{S}}, \bar{I}(\infty) = \hat{\bar{I}}, \tag{7.7}$$

Let us consider only (7.4) with (7.7) and the analysis of (7.4) with (7.6) is analogue. We linearize (7.4) about the point $(1-\frac{\mu_1+\beta-\gamma}{\mu},0,0)$ then determine the eigenvalues A, which are the roots of

$$\det\begin{pmatrix} \frac{(\mu_1+\beta-\mu-\gamma)}{c}-A & \frac{k(\mu-\mu_1-\beta+\gamma)}{c\mu}-\frac{\alpha}{c} & 0 \\ 0 & -A & 1 \\ \frac{-\beta}{d_2} & \frac{\mu(\mu_2+\alpha+\gamma)-k(\mu-\mu_1-\beta+\gamma)}{\mu d_2} & -\frac{c}{d_2}-A \end{pmatrix}=0 \tag{7.8}$$

Which are the roots of the characteristic polynomial,

$$P(A)=A^3+fA^2-gA+h,$$

with

$$f=\frac{c}{d_2}+\frac{(\mu+\gamma-\mu_1-\beta)}{c}, \tag{7.9}$$

$$g=\frac{(k-\mu)(\mu-\mu_1-\beta+\gamma)-\mu(\mu_2+\alpha+\gamma)}{\mu d_2},$$

$$h=\frac{(\mu_1+\beta-\mu-\gamma)^2+(\mu_1+\beta-\mu-\gamma)(\mu(\mu_2+\alpha+\gamma)-k\beta)-\alpha\beta\mu}{c\mu d_2}.$$

Using the theory of cubic polynomials we can easily see that it's possible to have three real roots.

To get stability to small linear perturbations we use the Routh-Hurwitz conditions for the roots P(A) to have negative real parts. This holds if

$$\begin{aligned} &\frac{c}{d_2}+\frac{(\mu+\gamma-\mu_1-\beta)}{c}>0, \\ &\frac{(\mu_1+\beta-\mu-\gamma)^2+(\mu_1+\beta-\mu-\gamma)(\mu(\mu_2+\alpha+\gamma)-k\beta)-\alpha\beta\mu}{c\mu d_2}>0 \\ &\left[\frac{(k-\mu)(\mu-\mu_1-\beta+\gamma)-\mu(\mu_2+\alpha+\gamma)}{\mu d_2}\right].\left[\frac{c}{d_2}+\frac{(\mu+\gamma-\mu_1-\beta)}{c}\right]-\frac{(\mu_1+\beta-\mu-\gamma)^2+(\mu_1+\beta-\mu-\gamma)(\mu(\mu_2+\alpha+\gamma)-k\beta)-\alpha\beta\mu}{c\mu d_2}<0. \end{aligned} \tag{7.10}$$

It is obvious that if the first and the second inequalities of (7.10) hold, the third equation of (7.10) does not hold unless $d_2<0$. Therefore in order to have a traveling wave

front solution which approach the steady state $(\hat{S},\hat{I},\hat{T})$ in an oscillatory manner as $z \to \infty$, we require that $d_2 < 0$ and (7.10) must hold, otherwise we have instability.

Conclusion

In this paper, we considered stability of the epidemic model SIQS We showed that if R_0 <1, the disease-free equilibrium is locally asymptotically stable, if R_0 =1, instable whereas if R_0 >1 the endemic equilibrium is locally asymptotically stable. Then we resolve the system SIQS epidemic model

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